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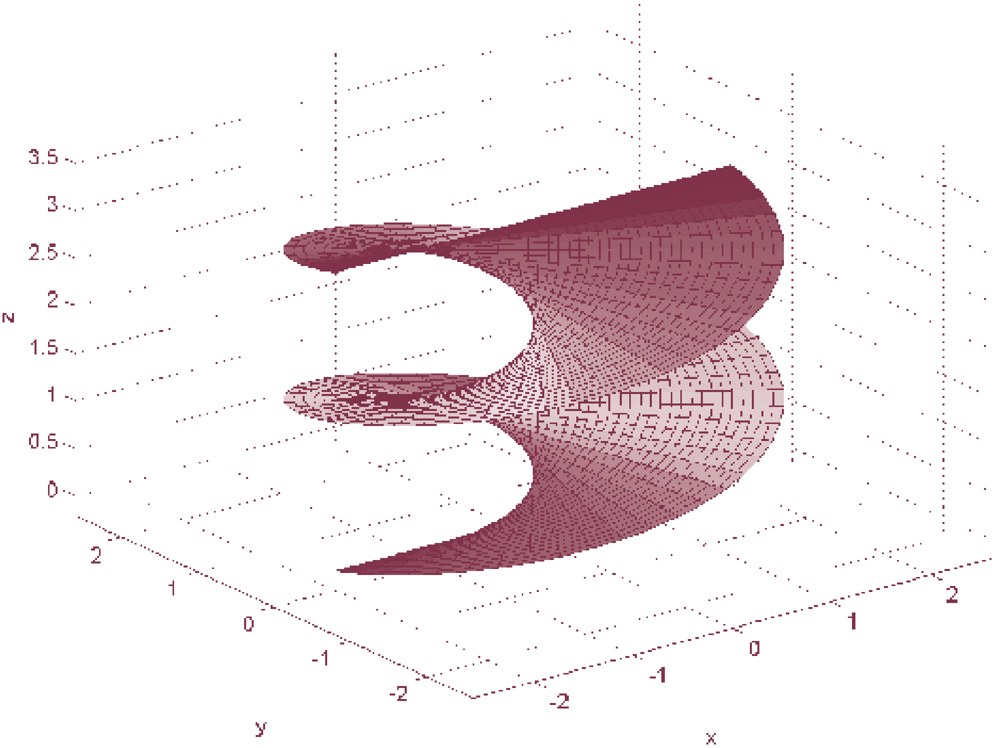

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Ostrowski type inequalities via $\psi-(\alpha,\beta,\gamma,\delta)-$convex function

Ali Hassan and Asif R. Khan

Abstract. In this paper, we are introducing very first time the class of $\psi-(\alpha,\beta,\gamma,\delta)-$convex function in mixed kind, which is the generalization of many classes of convex functions. We would like to state well-known Ostrowski inequality via Montgomery identity for $\psi-(\alpha,\beta,\gamma,\delta)-$convex function in mixed kind. In addition, we establish some Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $\psi-(\alpha,\beta,\gamma,\delta)$-convex functions in mixed kind by using different techniques including Hölder's inequality and power mean inequality. Also, various established results would be captured as special cases. Moreover, some applications in terms of special means would also be given.

Mathematics Subject Classification (2010): 26A33, 26A51, 26D15, 26D99, 47A30, 33B10.

Keywords: Ostrowski inequality, Montgomery identity, convex functions, special means.

1. Introduction

In almost every field of science, inequalities play a significant role. Although it is a very vast disciplineour focus is mainly on Ostrowski-type inequalities. In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives [15]. This inequality is well known in the literature as Ostrowski inequality.

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Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be differentiable function on (a, b) with the property that $|f'(t)| \leq M$ $\forall t \in (a, b)$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq M(b-a) \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right], \tag{1.1}$$

$\forall x \in (a, b)$. The constant $\frac{1}{4}$ is the best possible in the kind that it cannot be replaced by a smaller quantity.

Ostrowski inequality has applications in numerical integration, probability and optimization theory, statistics, information, and integral operator theory. Until now, a large number of research papers and books have been written on generalizations of Ostrowski inequalities and their numerous applications in [7]-[11]. Now we would like to present the Montgomery identity:

Theorem 1.2. [7] *Let $a < b$, $f \in AC[a, b]$ and $f' \in L_1[a, b]$, then the Montgomery identity holds:*

$$f(x) = \frac{1}{b-a} \int_a^b f(t)dt + \frac{1}{b-a} \int_a^b P_1(x, t) f'(t)dt,$$

where $P_1(x, t)$ is the Peano Kernel defined by:

$$P_1(x, t) = \begin{cases} t - a, & \text{if } t \in [a, x], \\ t - b, & \text{if } t \in (x, b], \end{cases}$$

$\forall x \in [a, b]$.

From literature, we recall and introduce some definitions for various convex functions.

Definition 1.3. [3] The $\tau : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex function, if

$$\tau\left(tx + (1-t)y\right) \leq t\tau(x) + (1-t)\tau(y),$$

$\forall x, y \in I, t \in [0, 1]$.

We recall here definition of $P-$convex function from [3]:

Definition 1.4. Let $\tau : I \subset \mathbb{R} \to \mathbb{R}$ is a $P-$convex, if $\tau(x) \geq 0$ and

$$\tau\left(tx + (1-t)y\right) \leq \tau(x) + \tau(y),$$

$\forall x, y \in I$ and $t \in [0, 1]$.

Here we also have definition of quasi$-$convex (for detailed discussion see [3].

Definition 1.5. The $\tau : I \subset \mathbb{R} \to \mathbb{R}$ is known as quasi$-$convex, if

$$\tau(tx + (1-t)y) \leq \max\{\tau(x), \tau(y)\}$$

$\forall x, y \in I,\ t \in [0, 1]$.

Now we present definition of $s-$convex functions in the first kind as follows which are extracted from [14]:

Definition 1.6. [4] Let $s \in (0,1]$. The $\tau : I \subset [0,\infty) \to [0,\infty)$ is said to be $s-$convex in the 1^{st} kind, if

$$\tau\left(tx+(1-t)y\right) \le t^s\tau(x) + (1-t^s)\tau(y),$$

$\forall x, y \in I, t \in [0,1]$.

Remark 1.7. If $s \to 0$, we get refinement of quasi-convexity (see Definition 1.5).

For second kind convexity we recall definition from [14].

Definition 1.8. Let $s \in (0,1]$. The $\tau : I \subset [0,\infty) \to [0,\infty)$ is said to be $s-$convex in the 2^{nd} kind, if

$$\tau\left(tx+(1-t)y\right) \le t^s\tau(x) + (1-t)^s\tau(y),$$

$\forall x, y \in I, t \in [0,1]$.

Remark 1.9. Further if $s \to 0$, we easily get $P-$convexity (see Definition 1.4).

Definition 1.10. [14] Let $(\alpha,\beta) \in (0,1]^2$. The $\tau : I \subset [0,\infty) \to [0,\infty)$ is said to be $(\alpha,\beta)-$convex in the 1^{st} kind, if

$$\tau\left(tx+(1-t)y\right) \le t^\alpha\tau(x) + (1-t^\beta)\tau(y),$$

$\forall x, y \in I, t \in [0,1]$.

Definition 1.11. [14] Let $(\alpha,\beta) \in (0,1]^2$. The $\tau : I \subset [0,\infty) \to [0,\infty)$ is said to be $(\alpha,\beta)-$convex in the 2^{nd} kind, if

$$\tau\left(tx+(1-t)y\right) \le t^\alpha\tau(x) + (1-t)^\beta\tau(y),$$

$\forall x, y \in I, t \in [0,1]$.

Definition 1.12. [14] The $\tau : I \subset \mathbb{R} \to \mathbb{R}$ is a Godunova-Levin convex, if $\tau(x) \ge 0$ and

$$\tau\left(tx+(1-t)y\right) \le \frac{1}{t}\tau(x) + \frac{1}{1-t}\tau(y),$$

$\forall x, y \in I$ and $t \in (0,1)$.

Definition 1.13. [14] The $\tau : I \subset \mathbb{R} \to [0,\infty)$ is of Godunova-Levin $s-$convex, with $s \in (0,1]$, if

$$\tau\left(tx+(1-t)y\right) \le \frac{1}{t^s}\tau(x) + \frac{1}{(1-t)^s}\tau(y),$$

$\forall t \in (0,1)$ and $x, y \in I$.

Definition 1.14. [14] Let $h : J \subseteq \mathbb{R} \to [0,\infty)$ with $h \neq 0$. The $\tau : I \subseteq \mathbb{R} \to [0,\infty)$ is an $h-$convex, if $\forall x, y \in I$, we have

$$\tau\left(tx+(1-t)y\right) \le h(t)\tau(x) + h(1-t)\tau(y),$$

$\forall t \in (0,1)$.

Definition 1.15. [3] The $\tau : I \subset \mathbb{R} \to \mathbb{R}$ is said to be $MT-$convex, if $\tau(x) \geq 0$, and

$$\tau\left(tx + (1-t)y\right) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}\tau(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}\tau(y),$$

$\forall x, y \in I, t \in (0,1)$.

Let $[a,b] \subseteq (0,+\infty)$, we may define special means as follows:

(a) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2};$$

(b) The geometric mean

$$G = G(a,b) := \sqrt{ab};$$

(c) The harmonic mean

$$H = H(a,b) := \frac{2}{\frac{1}{a} + \frac{1}{b}};$$

(d) The logarithmic mean

$$L = L(a,b) := \begin{cases} a & \text{if } a = b \\ \dfrac{b-a}{\ln b - \ln a}, & \text{if } a \neq b \end{cases};$$

(e) The identric mean

$$I = I(a,b) := \begin{cases} a & \text{if } a = b \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } a \neq b. \end{cases};$$

(f) The $p-$logarithmic mean

$$L_p = L_p(a,b) := \begin{cases} a & \text{if } a = b \\ \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if } a \neq b. \end{cases};$$

where $p \in \mathbb{R} \setminus \{0, -1\}$.

We make use of the beta function of Euler type, which is for $x, y > 0$ defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(x) = \int_0^\infty e^{-u}u^{x-1}du$.

The main aim of our study is to generalize the Ostrowski inequality (1.1) for $\psi - (\alpha, \beta, \gamma, \delta)-$convex in mixed kind, which is given in Section 2. Moreover, we establish some Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $\psi - (\alpha, \beta, \gamma, \delta)$-convex functions in mixed kind by using different techniques including Hölder's inequality and power means inequality. Also, we give special cases of our results. The application of midpoint inequalities in the special means, some particular cases of these inequalities are given in Section 3. The last section gives us a conclusion with some remarks and future ideas.

2. Generalization of Ostrowski type inequalities

Convexity is a very simple and ordinary concept, due to its massive applications in industry and business, convexity has a great influence on our daily life. In the solution of many real-world problems, the concept of convexity is very decisive. The problems faced in constrained control and estimation are convex. Geometrically, a real-valued function is said to be convex if the line segment joining any two of its points lies on or above the graph of the function in Euclidean space. We are introducing the very first time the class of $(s, r)-$convex and $\psi-(\alpha, \beta, \gamma, \delta)-$convex function in mixed kind.

Definition 2.1. [12] Let $(s, r) \in (0, 1]^2$. The $\tau : I \subset [0, \infty) \to [0, \infty)$ is said to be $(s, r)-$convex in mixed kind, if

$$\tau\left(tx + (1-t)y\right) \leq t^{rs}\tau(x) + (1-t^r)^s\tau(y),$$

$\forall x, y \in I, t \in [0, 1]$.

Definition 2.2. [12] Let $(\alpha, \beta, \gamma, \delta) \in (0, 1]^4$. The $\tau : I \subset [0, \infty) \to [0, \infty)$ is said to be $(\alpha, \beta, \gamma, \delta)-$convex in mixed kind, if

$$\tau\left(tx + (1-t)y\right) \leq t^{\alpha\gamma}\tau(x) + (1-t^\beta)^\delta\tau(y),$$

$\forall x, y \in I, t \in [0, 1]$.

Definition 2.3. [12] Let $\psi : (0, 1) \to (0, \infty)$, the $\tau : I \subset \mathbb{R} \to [0, \infty)$ is a $\psi-$convex, if $\forall x, y \in I$ we have

$$\tau\left(tx + (1-t)y\right) \leq t\psi(t)\tau(x) + (1-t)\psi(1-t)\tau(y),$$

$\forall t \in (0, 1)$.

Introducing a new class of convex functions that generalizes numerous well-known and highly regarded classes of convex functions, providing a broader framework for analysis and application in mathematical and optimization contexts.

Definition 2.4. Let $(\alpha, \beta, \gamma, \delta) \in (0, 1]^4$, and $\psi : (0, 1) \to (0, \infty)$. The $\tau : I \subset [0, \infty) \to [0, \infty)$ is said to be $\psi-(\alpha, \beta, \gamma, \delta)-$convex in mixed kind, if

$$\tau\left(tx + (1-t)y\right) \leq t^{\alpha\gamma}\psi(t)\tau(x) + (1-t^\beta)^\delta\psi(1-t)\tau(y), \tag{2.1}$$

$\forall x, y \in I, t \in [0, 1]$.

Remark 2.5. In Definition 2.4, we have the following cases.

1. If $\psi(t) = 1$ in (2.1), we get $(\alpha, \beta, \gamma, \delta)-$convex in mixed kind.
2. If $\psi(t) = \gamma = \delta = 1$ in (2.1), we get $(\alpha, \beta)-$convex in 1^{st} kind.
3. If $\psi(t) = \beta = \gamma = 1$ in (2.1), we get $(\alpha, \beta)-$convex in 2^{nd} kind.
4. If $\psi(t) = 1, \alpha = \delta = s, \beta = \gamma = r$, where $s, r \in (0, 1]$ in (2.1), we get $(s, r)-$convex in mixed kind.

5. If $\alpha = \beta = s$ and $\psi(t) = \gamma = \delta = 1$ where $s \in (0, 1]$ in (2.1), we get $s-$convex in 1^{st} kind.
6. If $\alpha = \beta \to 0$, and $\psi(t) = \gamma = \delta = 1$, in (2.1), we get refinement of quasi-convex.
7. If $\alpha = \delta = s$, $\psi(t) = \beta = \gamma = 1$ where $s \in (0, 1]$ or $(\alpha = \beta = \gamma = \delta = 1, \psi(t) = t^{s-1}$ with $s \in (0, 1])$ in (2.1), we get $s-$convex in 2^{nd} kind.
8. If $\alpha = \delta \to 0$, and $\psi(t) = \beta = \gamma = 1$, or $(\alpha = \beta = \gamma = \delta = 1$, and $\psi(t) = \frac{1}{t})$ in (2.1), we get $P-$convex.
9. If $\psi(t) = \alpha = \beta = \gamma = \delta = 1$ in (2.1), gives us ordinary convex.
10. If $\alpha = \beta = \gamma = \delta = 1$ in (2.1), gives us $\psi-$convex.
11. If $\alpha = \beta = \gamma = \delta = 1, l(t) = t$, $h = l\psi$ in (2.1), we get $h-$convex.
12. If $\alpha = \beta = \gamma = \delta = 1, \psi(t) = \frac{1}{t^{s+1}}$ with $s \in [0, 1)$ in (2.1), then we get the class of Godunova-Levin $s-$convex.
13. If $\alpha = \beta = \gamma = \delta = 1, \psi(t) = \frac{1}{t^2}$ in (2.1), then we get the concept of Godunova-Levin convex.
14. If $\alpha = \beta = \gamma = \delta = 1, \psi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (2.1), then we get the concept of $MT-$convex.

Theorem 2.6. *Suppose all the assumptions of Theorem* 1.2 *hold. If* $\tau : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ *is* $\psi - (\alpha, \beta, \gamma, \delta)-$*convex in mixed kind, then*

$$\begin{aligned}
&\tau\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right) \\
&\leq \left(\frac{x-a}{b-a}\right)^{\alpha\gamma} \psi\left(\frac{x-a}{b-a}\right)\left[\frac{1}{x-a}\int_a^x \tau\left[(t-a)f'(t)\right]dt\right] \\
&+ \left(1 - \left(\frac{x-a}{b-a}\right)^{\beta}\right)^{\delta} \psi\left(\frac{b-x}{b-a}\right)\left[\frac{1}{b-x}\int_x^b \tau\left[(t-b)f'(t)\right]dt\right], \qquad (2.2)
\end{aligned}$$

$\forall x \in [a, b]$.

Proof. Utilizing the Theorem 1.2, we get

$$\begin{aligned}
f(x) - \frac{1}{b-a}\int_a^b f(t)dt &= \left(\frac{x-a}{b-a}\right)\left[\frac{1}{x-a}\int_a^x (t-a)f'(t)dt\right] \\
&+ \left(1 - \left(\frac{x-a}{b-a}\right)\right)\left[\frac{1}{b-x}\int_x^b (t-b)f'(t)dt\right],
\end{aligned}$$

using the $\psi - (\alpha, \beta, \gamma, \delta)-$convexity in mixed kind of $\tau : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$, we have

$$\begin{aligned}
&\tau\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right) \\
&\leq \left(\frac{x-a}{b-a}\right)^{\alpha\gamma} \psi\left(\frac{x-a}{b-a}\right)\tau\left[\frac{1}{x-a}\int_a^x (t-a)f'(t)dt\right] \\
&+ \left(1 - \left(\frac{x-a}{b-a}\right)^{\beta}\right)^{\delta} \psi\left(\frac{b-x}{b-a}\right)\tau\left[\frac{1}{b-x}\int_x^b (t-b)f'(t)dt\right],
\end{aligned}$$

$\forall x \in [a,b]$, which is an inequality of interest in itself as well. If we use Jensen's integral inequality we get (2.2). $\square$

Corollary 2.7. *In Theorem* 2.6, *one can see the following.*

1. *If* $\psi(t) = 1$, *in* (2.2), *then functional generalization of Ostrowski inequality for* $(\alpha,\beta,\gamma,\delta)-$*convex functions in mixed kind:*

$$\begin{aligned}&\tau\left(f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right)\\ &\leq \left(\frac{x-a}{b-a}\right)^{\alpha\gamma}\left[\frac{1}{x-a}\int_a^x \tau\left[(t-a)f'(t)\right]dt\right]\\ &+\left(1-\left(\frac{x-a}{b-a}\right)^{\beta}\right)^{\delta}\left[\frac{1}{b-x}\int_x^b \tau\left[(t-b)f'(t)\right]dt\right]\end{aligned}$$

2. *If* $\psi(t) = \gamma = \delta = 1$, *and* $\alpha,\beta \in (0,1]$ *in* (2.2), *then functional generalization of Ostrowski inequality for* $(\alpha,\beta)-$*convex functions in* 1^{st} *kind:*

$$\begin{aligned}&\tau\left(f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right)\\ &\leq \left(\frac{x-a}{b-a}\right)^{\alpha}\left[\frac{1}{x-a}\int_a^x \tau[(t-a)f'(t)]dt\right]\\ &+\left(1-\left(\frac{x-a}{b-a}\right)^{\beta}\right)\left[\frac{1}{b-x}\int_x^b \tau[(t-b)f'(t)]dt\right].\end{aligned}$$

3. *If* $\psi(t) = \beta = \gamma = 1$, *and* $\alpha,\delta \in (0,1]$ *in* (2.2), *then functional generalization of Ostrowski inequality for* $(\alpha,\delta)-$*convex functions in* 2^{nd} *kind:*

$$\begin{aligned}&\tau\left(f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right)\\ &\leq \left(\frac{x-a}{b-a}\right)^{\alpha}\left[\frac{1}{x-a}\int_a^x \tau[(t-a)f'(t)]dt\right]\\ &+\left(1-\left(\frac{x-a}{b-a}\right)\right)^{\delta}\left[\frac{1}{b-x}\int_x^b \tau[(t-b)f'(t)]dt\right].\end{aligned}$$

4. *If* $\psi(t) = 1, \alpha = \delta = s$, *and* $\beta = \gamma = r$, *where* $s,r \in (0,1]$ *in* (2.2), *then functional generalization of Ostrowski inequality for* $(s,r)-$*convex functions in mixed kind:*

$$\begin{aligned}&\tau\left(f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right)\\ &\leq \left(\frac{x-a}{b-a}\right)^{rs}\left[\frac{1}{x-a}\int_a^x \tau[(t-a)f'(t)]dt\right]\\ &+\left(1-\left(\frac{x-a}{b-a}\right)^{r}\right)^{s}\left[\frac{1}{b-x}\int_x^b \tau[(t-b)f'(t)]dt\right].\end{aligned}$$

5. *If* $\alpha = \beta = s$ *and* $\psi(t) = \gamma = \delta = 1$, *where* $s \in (0,1]$ *in* (2.2), *then functional generalization of Ostrowski inequality for* $s-$*convex functions in* 1^{st} *kind:*

$$\begin{aligned}
&\tau\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right) \\
&\leq \left(\frac{x-a}{b-a}\right)^s \left[\frac{1}{x-a}\int_a^x \tau[(t-a)f'(t)]dt\right] \\
&+ \left(1 - \left(\frac{x-a}{b-a}\right)^s\right) \left[\frac{1}{b-x}\int_x^b \tau[(t-b)f'(t)]dt\right].
\end{aligned}$$

6. *If* $\alpha = \beta \to 0$ *and* $\psi(t) = \gamma = \delta = 1$ *in* (2.2), *then functional generalization of Ostrowski inequality for quasi*$-$*convex functions:*

$$\tau\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right) \leq \frac{1}{x-a}\int_a^x \tau[(t-a)f'(t)]dt.$$

7. *If* $\alpha = \delta = s$, *and* $\psi(t) = \beta = \gamma = 1$, *where* $s \in [0,1]$ *in* (2.2), *then functional generalization of Ostrowski inequality for* $s-$*convex functions in* 2^{nd} *kind:*

$$\begin{aligned}
&\tau\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right) \\
&\leq \left(\frac{x-a}{b-a}\right)^s \left[\frac{1}{x-a}\int_a^x \tau[(t-a)f'(t)]dt\right] \\
&+ \left(\frac{b-x}{b-a}\right)^s \left[\frac{1}{b-x}\int_x^b \tau[(t-b)f'(t)]dt\right].
\end{aligned}$$

8. *If* $\alpha = \delta \to 0$ *and* $\psi(t) = \beta = \gamma = 1$ *in* (2.2), *then functional generalization of Ostrowski inequality for* $P-$*convex functions:*

$$\begin{aligned}
&\tau\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right) \\
&\leq \frac{1}{x-a}\int_a^x \tau[(t-a)f'(t)]dt + \frac{1}{b-x}\int_x^b \tau[(t-b)f'(t)]dt.
\end{aligned}$$

9. *If* $\psi(t) = \alpha = \beta = \gamma = \delta = 1$ *in* (2.2), *then functional generalization of Ostrowski inequality for convex functions which is inequality* (2.1) *of Theorem* 7 *in* [8].

10. *If* $\alpha = \beta = \gamma = \delta = 1$, *in* (2.2), *then functional generalization of Ostrowski inequality for* $\psi-$*convex functions:*

$$\begin{aligned}
&\tau\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right) \\
&\leq \frac{1}{b-a}\left[\psi\left(\frac{x-a}{b-a}\right)\int_a^x \tau\left[(t-a)\, f'(t)\right]dt\right. \\
&\left.+\psi\left(\frac{b-x}{b-a}\right)\int_x^b \tau\left[(t-b)\, f'(t)\right]dt\right].
\end{aligned}$$

11. *If* $\alpha = \beta = \gamma = \delta = 1, l(t) = t$, *and* $h = l\psi$ *in* (2.2), *then functional generalization of Ostrowski inequality for* $h-$*convex functions:*

$$\begin{aligned}
&\tau\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right) \\
&\leq h\left(\frac{x-a}{b-a}\right)\left[\frac{1}{x-a}\int_a^x \tau[(t-a)f'(t)]dt\right] \\
&+h\left(\frac{b-x}{b-a}\right)\left[\frac{1}{b-x}\int_x^b \tau[(t-b)f'(t)]dt\right].
\end{aligned}$$

12. *If* $\alpha = \beta = \gamma = \delta = 1, \psi(t) = \frac{1}{t^{s+1}}$ *with* $s \in [0,1]$ *in* (2.2), *then functional generalization of Ostrowski inequality for GL* $s-$*convex:*

$$\begin{aligned}
&\tau\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right) \\
&\leq (b-a)^s\left[\frac{1}{(x-a)^{s+1}}\int_a^x \tau[(t-a)f'(t)]dt\right. \\
&\left.+\frac{1}{(b-x)^{s+1}}\int_x^b \tau[(t-b)f'(t)]dt\right].
\end{aligned}$$

13. *If* $\alpha = \beta = \gamma = \delta = 1, \psi(t) = \frac{1}{t^2}$ *in* (2.2), *then functional generalization of Ostrowski inequality for GL convex:*

$$\begin{aligned}
&\tau\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right) \\
&\leq (b-a)\left[\frac{1}{(x-a)^2}\int_a^x \tau[(t-a)f'(t)]dt + \frac{1}{(b-x)^2}\int_x^b \tau[(t-b)f'(t)]dt\right].
\end{aligned}$$

14. If $\alpha=\beta=\gamma=\delta=1, \psi(t)=\frac{1}{2\sqrt{t(1-t)}}$ *in* (2.2), *then functional generalization of Ostrowski inequality for* $MT-$*convex:*

$$\begin{aligned}&\tau\left(f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right)\\&\leq \frac{1}{2\sqrt{(x-a)(b-x)}}\left[\int_a^x \tau[(t-a)f'(t)]dt+\int_x^b \tau[(t-b)f'(t)]dt\right].\end{aligned}$$

In order to prove our main results, we need the following lemma that has been obtained in [16].

Lemma 2.8. *Let* $f:[a,b]\to\mathbb{R}$ *be an absolutely continuous mapping on* (a,b) *with* $a<b$. *If* $f'\in L_1[a,b]$, *then* $\forall x\in(a,b)$

$$\begin{aligned}f(x)-\frac{1}{b-a}\int_a^b f(t)dt &= \frac{(x-a)^2}{b-a}\int_0^1 tf'(tx+(1-t)a)dt\\&\quad-\frac{(b-x)^2}{b-a}\int_0^1 tf'(tx+(1-t)b)dt.\end{aligned}$$

Theorem 2.9. *Let* $a<b, f\in AC[a,b], f'\in L_1[a,b]$, *and* $|f'|$ *is* $\psi-(\alpha,\beta,\gamma,\delta)-$*convex function with* $|f'(x)|\leq M$, *then* $\forall x\in(a,b)$

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\\&\leq M\left(\int_0^1\left(t^{\alpha\gamma+1}\psi(t)+t(1-t^{\beta})^{\delta}\psi(1-t)\right)dt\right)\ \kappa_a^b(x), \qquad (2.3)\end{aligned}$$

where $\kappa_a^b(x)=\frac{(x-a)^2+(b-x)^2}{b-a}$.

Proof. From the Lemma 2.8 we have

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| &\leq \frac{(x-a)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)a)\right|dt\\&\quad+\frac{(b-x)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)b)\right|dt. \qquad (2.4)\end{aligned}$$

Since $|f'|$ is $\psi-(\alpha,\beta,\gamma,\delta)-$convex and $|f'(x)|\leq M$, we get

$$\int_0^1 t\left|f'(tx+(1-t)a)\right|dt\leq M\int_0^1 t\left(t^{\alpha\gamma}\psi(t)+(1-t^{\beta})^{\delta}\psi(1-t)\right)dt, \qquad (2.5)$$

and similarly

$$\int_0^1 t\left|f'(tx+(1-t)b)\right|dt\leq M\int_0^1 t\left(t^{\alpha\gamma}\psi(t)+(1-t^{\beta})^{\delta}\psi(1-t)\right)dt. \qquad (2.6)$$

By using inequalities (2.5) and (2.6) in (2.4), we get (2.3). □

Corollary 2.10. *In Theorem* 2.9, *one can see the following.*

1. *If $\psi(t)=1$, in (2.3), then Ostrowski inequality for $(\alpha,\beta,\gamma,\delta)-$convex functions in mixed kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq M\left(\frac{1}{\alpha\gamma+2}+\frac{B\left(\frac{2}{\beta},\delta+1\right)}{\beta}\right)\ \kappa_a^b(x).$$

2. *If $\psi(t)=\gamma=\delta=1$, $\alpha\in[0,1]$ and $\beta\in(0,1]$, in (2.3), then Ostrowski inequality for $(\alpha,\beta)-$convex functions in 1^{st} kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq M\left(\frac{1}{\alpha+2}+\frac{B\left(\frac{2}{\beta},2\right)}{\beta}\right)\ \kappa_a^b(x).$$

3. *If $\psi(t)=\beta=\gamma=1$, $\alpha\in[0,1]$ and $\delta\in[0,1]$, in (2.3), then Ostrowski inequality for $(\alpha,\delta)-$convex functions in 2^{nd} kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq M\left(\frac{1}{\alpha+2}+\frac{1}{(\delta+1)(\delta+2)}\right)\ \kappa_a^b(x).$$

4. *If $\psi(t)=1,\alpha=\delta=s$, $\beta=\gamma=r$, where $s\in[0,1]$ and $r\in(0,1]$ in (2.3), then Ostrowski inequality for $(s,r)-$convex functions in mixed kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq M\left(\frac{1}{rs+2}+\frac{B\left(\frac{2}{r},s+1\right)}{r}\right)\ \kappa_a^b(x).$$

5. *If $\alpha=\beta=s$ and $\psi(t)=\gamma=\delta=1$, where $s\in(0,1]$ in (2.3), then Ostrowski inequality for $s-$convex functions in 1^{st} kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq M\left(\frac{1}{s+2}+\frac{B\left(\frac{2}{s},2\right)}{s}\right)\ \kappa_a^b(x).$$

6. *If $\alpha=\delta\to 0$ and $\psi(t)=\beta=\gamma=1$ in (2.3), then Ostrowski inequality for $P-$convex functions:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq M\ \kappa_a^b(x).$$

7. *If $\psi(t)=\beta=\gamma=1$, $\alpha=\delta=s$ where $s\in[0,1]$, then (2.3) reduces to the inequality (2.1) of Theorem 2 in [1].*
8. *If $\psi(t)=\alpha=\beta=\gamma=\delta=1$, then (2.3) reduces to the inequality (1.1).*
9. *If $\alpha=\beta=\gamma=\delta=1$ in (2.3), then Ostrowski inequality for $\psi-$convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq M\left(\int_0^1\left(t^2\psi(t)+t(1-t)\psi(1-t)\right)dt\right)\ \kappa_a^b(x).$$

10. *If $\alpha=\beta=\gamma=\delta=1,l(t)=t$, then if $h=l\psi$, in (2.3), then Ostrowski inequality for $h-$convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq M\left(\int_0^1\left(th(t)+th(1-t)\right)dt\right)\ \kappa_a^b(x).$$

11. *If* $\alpha=\beta=\gamma=\delta=1, \psi(t)=t^{-(s+1)}$ *in* (2.3), *then Ostrowski inequality for GL* $s-$*convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq M\left(\frac{1}{1-s}\right)\ \kappa_a^b(x).$$

12. *If* $\alpha=\beta=\gamma=\delta=1, \psi(t)=\frac{1}{2\sqrt{t(1-t)}}$ *in* (2.3), *then Ostrowski inequality for* $MT-$*convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M\pi}{4}\ \kappa_a^b(x).$$

Theorem 2.11. *Let* $a<b, f\in AC[a,b], f'\in L_1[a,b]$, *and* $|f'|^q$ *is* $\psi-(\alpha,\beta,\gamma,\delta)-$*convex function for* $q\geq 1$ *with* $|f'(x)|\leq M$, *then* $\forall x\in(a,b)$

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{M}{(2)^{1-\frac{1}{q}}}\left(\int_0^1\left(t^{\alpha\gamma+1}\psi(t)+t(1-t^\beta)^\delta\psi(1-t)\right)dt\right)^{\frac{1}{q}}\ \kappa_a^b(x). \tag{2.7}$$

Proof. From the Lemma 2.8 and power mean inequality, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(x-a)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(tx+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(tx+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}. \tag{2.8}$$

Since $|f'|^q$ is $\psi-(\alpha,\beta,\gamma,\delta)-$convex and $|f'(x)|\leq M$, we get

$$\int_0^1 t\left|f'\left(tx+(1-t)a\right)\right|^q dt \leq M^q\int_0^1 t\left(t^{\alpha\gamma}\psi(t)+(1-t^\beta)^\delta\psi(1-t)\right)dt, \tag{2.9}$$

and

$$\int_0^1 t\left|f'\left(tx+(1-t)b\right)\right|^q dt \leq M^q\int_0^1 t\left(t^{\alpha\gamma}\psi(t)+(1-t^\beta)^\delta\psi(1-t)\right)dt. \tag{2.10}$$

Using the inequalities (2.8) – (2.10), we get (2.7). ☐

Corollary 2.12. *In Theorem* 2.11, *one can see the following.*

1. *If* $q=1$, *then we get Theorem* 2.9.
2. *If* $\psi(t)=1$, *in* (2.7), *then Ostrowski inequality for* $(\alpha,\beta,\gamma,\delta)-$*convex functions in mixed kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{\alpha\gamma+2}+\frac{B\left(\frac{2}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}\ \kappa_a^b(x).$$

3. *If $\psi(t)=\gamma=\delta=1$, $\alpha\in[0,1]$ and $\beta\in(0,1]$, in (2.7), then Ostrowski inequality for $(\alpha,\beta)-$convex functions in 1^{st} kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{\alpha+2}+\frac{B\left(\frac{2}{\beta},2\right)}{\beta}\right)^{\frac{1}{q}}\ \kappa_a^b(x).$$

4. *If $\psi(t)=\beta=\gamma=1$, $\alpha\in[0,1]$ and $\delta\in[0,1]$, in (2.7), then Ostrowski inequality for $(\alpha,\delta)-$convex functions in 2^{nd} kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{(\alpha+2)}+\frac{1}{(\delta+1)(\delta+2)}\right)^{\frac{1}{q}}\ \kappa_a^b(x).$$

5. *If $\psi(t)=1,\alpha=\delta=s$, $\beta=\gamma=r$, where $s\in[0,1]$ and $r\in(0,1]$ in (2.7), then Ostrowski inequality for $(s,r)-$convex functions in mixed kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{rs+2}+\frac{B\left(\frac{2}{r},s+1\right)}{r}\right)^{\frac{1}{q}}\ \kappa_a^b(x).$$

6. *If $\alpha=\beta=s$ and $\psi(t)=\gamma=\delta=1$, where $s\in(0,1]$ in (2.7), then Ostrowski inequality for $s-$convex functions in 1^{st} kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{s+2}+\frac{B\left(\frac{2}{s},2\right)}{s}\right)^{\frac{1}{q}}\ \kappa_a^b(x).$$

7. *If $\alpha=\delta\to0$ and $\psi(t)=\beta=\gamma=1$ in (2.7), then Ostrowski inequality for $P-$convex functions:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{(2)^{1-\frac{1}{q}}}\ \kappa_a^b(x).$$

8. *If $\psi(t)=\beta=\gamma=1$, $\alpha=\delta=s$ where $s\in[0,1]$, then (2.7) reduces to the inequality (2.3) of Theorem 4 in [1].*
9. *If $\psi(t)=\alpha=\beta=\gamma=\delta=1$, then (2.7) reduces to the inequality (1.1).*
10. *If $\alpha=\beta=\gamma=\delta=1$, in (2.7), then Ostrowski inequality for $\psi-$convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq\frac{M}{2^{1-\frac{1}{q}}}\left(\int_0^1\left(t^2\psi(t)+t(1-t)\psi(1-t)\right)dt\right)^{\frac{1}{q}}\ \kappa_a^b(x).$$

11. *If $\alpha=\beta=\gamma=\delta=1,l(t)=t$, then if $h=l\psi$, in (2.7), then Ostrowski inequality for $h-$convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{2^{1-\frac{1}{q}}}\left(\int_0^1\left(th(t)+th(1-t)\right)dt\right)^{\frac{1}{q}}\ \kappa_a^b(x).$$

12. If $\alpha=\beta=\gamma=\delta=1, \psi(t)=t^{-(s+1)}$ *in* (2.7), *then Ostrowski inequality for GL* $s-$*convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{M}{2^{1-\frac{1}{q}}}\left(\frac{1}{1-s}\right)^{\frac{1}{q}} \kappa_a^b(x).$$

13. If $\alpha=\beta=\gamma=\delta=1, \psi(t)=\frac{1}{2\sqrt{t(1-t)}}$ *in* (2.7), *then Ostrowski inequality for* $MT-$*convex:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{M\tau^{\frac{1}{q}}}{2^{1+\frac{1}{q}}} \kappa_a^b(x).$$

Theorem 2.13. *Let* $a<b, f\in AC[a,b], f'\in L_1[a,b]$, *and* $|f'|^q$ *is* $\psi-(\alpha,\beta,\gamma,\delta)-$*convex function for* $q>1$ *with* $|f'(x)|\le M$, *then* $\forall x\in(a,b)$

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{M}{(p+1)^{\frac{1}{p}}}\left(\int_0^1 \left(t^{\alpha\gamma}\psi(t)+(1-t^{\beta})^{\delta}\psi(1-t)\right)dt\right)^{\frac{1}{q}} \kappa_a^b(x), \qquad (2.11)$$

where $p^{-1}+q^{-1}=1$.

Proof. From the Lemma 2.8 and Hölder's inequality, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{(x-a)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(tx+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} + \frac{(b-x)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(tx+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}. \qquad (2.12)$$

Since $|f'|^q$ is $\psi-(\alpha,\beta,\gamma,\delta)-$convex and $|f'(x)|\le M$, we get

$$\int_0^1 \left|f'\left(tx+(1-t)a\right)\right|^q dt \le M^q\int_0^1 \left(t^{\alpha\gamma}\psi(t)+(1-t^{\beta})^{\delta}\psi(1-t)\right)dt, \qquad (2.13)$$

and

$$\int_0^1 \left|f'\left(tx+(1-t)b\right)\right|^q dt \le M^q\int_0^1 \left(t^{\alpha\gamma}\psi(t)+(1-t^{\beta})^{\delta}\psi(1-t)\right)dt. \qquad (2.14)$$

Using inequalities (2.12) – (2.14), we get (2.11). □

Corollary 2.14. *In Theorem* 2.13, *one can see the following.*

1. If $\psi(t)=1$, *in* (2.11), *then Ostrowski inequality for* $(\alpha,\beta,\gamma,\delta)-$*convex in mixed kind:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha\gamma+1}+\frac{B\left(\frac{1}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}} \kappa_a^b(x).$$

2. *If* $\psi(t) = \gamma = \delta = 1$, $\alpha \in [0,1]$ *and* $\beta \in (0,1]$, *in* (2.11), *then Ostrowski inequality for* $(\alpha,\beta)-$*convex in* 1^{st} *kind:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha+1} + \frac{B\left(\frac{1}{\beta},2\right)}{\beta}\right)^{\frac{1}{q}} \kappa_a^b(x).$$

3. *If* $\psi(t) = \beta = \gamma = 1$, $\alpha \in [0,1]$ *and* $\delta \in [0,1]$, *in* (2.11), *then Ostrowski inequality for* $(\alpha,\delta)-$*convex in* 2^{nd} *kind:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha+1} + \frac{1}{\delta+1}\right)^{\frac{1}{q}} \kappa_a^b(x).$$

4. *If* $\psi(t) = 1, \alpha = \delta = s$, $\beta = \gamma = r$, *where* $s \in [0,1]$ *and* $r \in (0,1]$ *in* (2.11), *then Ostrowski inequality for* $(s,r)-$*convex in mixed kind:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{rs+1} + \frac{B\left(\frac{1}{r},s+1\right)}{r}\right)^{\frac{1}{q}} \kappa_a^b(x).$$

5. *If* $\alpha = \beta = s$ *and* $\psi(t) = \gamma = \delta = 1$, *where* $s \in (0,1]$ *in* (2.11), *then Ostrowski inequality for* $s-$*convex in* 1^{st} *kind:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{s+1} + \frac{B\left(\frac{1}{s},2\right)}{s}\right)^{\frac{1}{q}} \kappa_a^b(x).$$

6. *If* $\psi(t) = \beta = \gamma = 1$, $\alpha = \delta = s$ *where* $s \in [0,1]$, *then* (2.11) *reduces to the inequality* (2.2) *of Theorem* 3 *in* [1].
7. *If* $\alpha = \delta \to 0$ *and* $\psi(t) = \beta = \gamma = 1$ *in* (2.11), *then Ostrowski inequality for* $P-$*convex:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(2)^{\frac{1}{q}}M}{(p+1)^{\frac{1}{p}}} \kappa_a^b(x).$$

8. *If* $\psi(t) = \alpha = \beta = \gamma = \delta = 1$ *in* (2.11), *then Ostrowski inequality for convex:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{(p+1)^{\frac{1}{p}}} \kappa_a^b(x).$$

9. *If* $\alpha = \beta = \gamma = \delta = 1$, *in* (2.11), *then Ostrowski inequality for* $\psi-$*convex:*

$$\begin{aligned}&\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \\ &\leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\int_0^1 (t\psi(t) + (1-t)\psi(1-t))\,dt\right)^{\frac{1}{q}} \kappa_a^b(x).\end{aligned}$$

10. *If* $\alpha = \beta = \gamma = \delta = 1, l(t) = t$, *then if* $h = l\psi$, *in* (2.11), *then Ostrowski inequality for* $h-$*convex:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\int_0^1 (h(t) + h(1-t))\,dt\right)^{\frac{1}{q}} \kappa_a^b(x).$$

11. *If* $\alpha = \beta = \gamma = \delta = 1, \psi(t) = t^{-(s+1)}$ *where* $s \in [0,1)$ *in* (2.11), *then Ostrowski inequality for* $GL\ s-$*convex:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{2}{1-s}\right)^{\frac{1}{q}} \kappa_a^b(x).$$

12. *If* $\alpha = \beta = \gamma = \delta = 1, \psi(t) = \frac{1}{2\sqrt{t(1-t)}}$ *in* (2.11), *then Ostrowski inequality for* $MT-$*convex:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M\left(\frac{\pi}{2}\right)^{\frac{1}{q}}}{(1+p)^{\frac{1}{p}}} \kappa_a^b(x).$$

3. Applications of midpoint Ostrowski type inequalities via $\psi-(\alpha,\beta,\gamma,\delta)-$convex

If we replace f by $-f$ and $x = \frac{a+b}{2}$ in Theorem 2.6, then the functional generalization of Ostrowski midpoint inequality for $\psi-(\alpha,\beta,\gamma,\delta)-$ convex functions:

$$\begin{aligned}&\tau\left(\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)\right)\\ &\leq \frac{\psi\left(\frac{1}{2}\right)}{b-a}\left[\frac{1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}} \tau[(a-t)f'(t)]dt + \frac{\left(2^\beta-1\right)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^b \tau[(b-t)f'(t)]dt\right].\end{aligned} \tag{3.1}$$

Remark 3.1. Assume that $\tau : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be an $\psi-(\alpha,\beta,\gamma,\delta)-$convex function in mixed kind:

1. If $f(t) = \frac{1}{t}$ in inequality (3.1) where $t \in [a,b] \subset (0,\infty)$, then

$$\begin{aligned}&\tau\left[\frac{A(a,b)-L(a,b)}{A(a,b)L(a,b)}\right]\\ &\leq \frac{\psi\left(\frac{1}{2}\right)}{b-a}\left[\frac{1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}} \tau\left[\frac{t-a}{t^2}\right]dt + \frac{\left(2^\beta-1\right)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^b \tau\left[\frac{t-b}{t^2}\right]dt\right].\end{aligned}$$

2. If $f(t) = -\ln t$ in inequality (3.1), where $t \in [a,b] \subset (0,\infty)$, then

$$\begin{aligned}&\tau\left[\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right]\\ &\leq \frac{\psi\left(\frac{1}{2}\right)}{b-a}\left[\frac{1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}} \tau\left[\frac{t-a}{t}\right]dt + \frac{\left(2^\beta-1\right)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^b \tau\left[\frac{t-b}{t}\right]dt\right].\end{aligned}$$

3. If $f(t) = t^p, p \in \mathbb{R} \setminus \{0, -1\}$ in inequality (3.1), where $t \in [a,b] \subset (0,\infty)$, then

$$\begin{aligned}\tau\left[L_p^p(a,b) - A^p(a,b)\right] \leq \frac{\psi\left(\frac{1}{2}\right)}{b-a}&\left[\frac{1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}} \tau\left[\frac{p\,(a-t)}{t^{1-p}}\right]dt\right.\\ &\left.+\frac{\left(2^\beta-1\right)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^b \tau\left[\frac{p\,(b-t)}{t^{1-p}}\right]dt\right].\end{aligned}$$

Remark 3.2. In Theorem 2.11, one can see the following.

1. Let $x = \frac{a+b}{2}$, $0 < a < b$, $q \geq 1$ and $f : \mathbb{R} \to \mathbb{R}^+$, $f(t) = t^n$ in (2.7). Then

$$\begin{aligned}&\left|A^n(a,b) - L_n^n(a,b)\right|\\ &\leq \frac{M\,(b-a)}{(2)^{2-\frac{1}{q}}}\left(\int_0^1 \left(t^{\alpha\gamma+1}\psi(t) + t(1-t^\beta)^\delta\psi(1-t)\right)dt\right)^{\frac{1}{q}}.\end{aligned}$$

2. Let $x = \frac{a+b}{2}$, $0 < a < b$, $q \geq 1$ and $f : (0,1] \to \mathbb{R}$, $f(t) = -\ln t$ in (2.7). Then

$$\left|\ln\left(\frac{A\,(a,b)}{I\,(a,b)}\right)\right| \leq \frac{M\,(b-a)}{(2)^{2-\frac{1}{q}}}\left(\int_0^1 \left(t^{\alpha\gamma+1}\psi(t) + t(1-t^\beta)^\delta\psi(1-t)\right)dt\right)^{\frac{1}{q}}.$$

Remark 3.3. In Theorem 2.13, one can see the following.

1. Let $x = \frac{a+b}{2}$, $0 < a < b$, $p^{-1} + q^{-1} = 1$ and $f : \mathbb{R} \to \mathbb{R}^+$, $f(t) = t^n$ in (2.11). Then

$$\begin{aligned}&\left|A^n(a,b) - L_n^n(a,b)\right|\\ &\leq \frac{M\,(b-a)}{2\,(p+1)^{\frac{1}{p}}}\left(\int_0^1 \left(t^{\alpha\gamma}\psi(t) + (1-t^\beta)^\delta\psi(1-t)\right)dt\right)^{\frac{1}{q}}.\end{aligned}$$

2. Let $x = \frac{a+b}{2}$, $0 < a < b$, $p^{-1} + q^{-1} = 1$ and $f : (0,1] \to \mathbb{R}$, $f(t) = -\ln t$ in (2.11). Then

$$\left|\ln\left(\frac{A\,(a,b)}{I\,(a,b)}\right)\right| \leq \frac{M\,(b-a)}{2\,(p+1)^{\frac{1}{p}}}\left(\int_0^1 \left(t^{\alpha\gamma}\psi(t) + (1-t^\beta)^\delta\psi(1-t)\right)dt\right)^{\frac{1}{q}}.$$

4. Conclusion and remarks

4.1. Conclusion

Ostrowski inequality is one of the most celebrated inequalities, we can find its various generalizations and variants in literature. In this paper, we presented the generalized notion of $\psi - (\alpha, \beta, \gamma, \delta)-$convex functions in mixed kind. This class of functions contains many important classes. We have started our first main result in section 2, the generalization of Ostrowski inequality via Montgomery identity with $\psi - (\alpha, \beta, \gamma, \delta)-$convex functions in mixed kind. Further, we used different techniques including Hölder's inequality and power mean inequality for generalization of Ostrowski inequality[15]. Finally, we have given some applications in terms of special means including arithmetic, geometric, harmonic, logarithmic, identric, and $p-$logarithmic means by using the midpoint inequalities.

4.2. Remarks and future ideas

1. One may also do similar work by using various different classes of convex functions.
2. One may do similar work to generalize all results stated in this research work by applying weights.
3. One may also state all results stated in this research work by higher-order derivatives.
4. One may also state all results stated in this research work by multivariable functions.
5. One may try to state all results stated in this research work for generalized fractional integral operators.
6. One may try to state all results stated in this research work for Jensen-Steffensen inequality and their different types of variants.
7. One may also do the similar work by using various different generalized forms for the Korkine's and Montgomery identities, improved power means inequality, Hölder's Iscan inequality, Jensen's integral inequality with weights, generalized fuzzy metric spaces on the set of all fuzzy numbers.

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Better approximations for quasi-convex functions

Huriye Kadakal

Abstract. In this paper, by using Hölder-İşcan, Hölder integral inequality and an general identity for differentiable functions we can get new estimates on generalization of Hadamard, Ostrowski and Simpson type integral inequalities for functions whose derivatives in absolute value at certain power are quasi-convex functions. It is proved that the result obtained Hölder-İşcan integral inequality is better than the result obtained Hölder inequality.

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1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f\left(tx + (1-t)y\right) \leq tf\left(x\right) + (1-t)f\left(y\right)$$

valid for all $x, y \in I$ and $t \in [0, 1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \varnothing$.

Integral inequalities have played an important role in the development of all branches of Mathematics and the other sciences. The inequalities discovered by Hermite and Hadamard for convex functions are very important in the literature. The classical Hermite-Hadamard integral inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$. Firstly, let's recall the Hermite-Hadamard integral inequality. In addition, readers can refer to the [8, 9, 10, 11, 14, 16, 12, 13, 17, 18, 19] articles and the references therein for more detailed information on both convexity and the different classes of convexity.

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Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions [1, 4]. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if the function f is concave.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping differentiable in I°, the interior of I, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$, $x \in [a, b]$, then we the following inequality holds

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{M}{b-a}\left[\frac{(x-a)^2 + (b-x)^2}{2}\right]$$

for all $x \in [a, b]$. This result is known in the literature as the Ostrowski inequality [3].

The following inequality is well known in the literature as Simpson's inequality

.

Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then the following inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

In recent years many authors have studied error estimations for Simpson's inequality; for refinements, counterparts, generalizations and new Simpson's type inequalities, see [20, 21] and therein.

Definition 1.1 ([2]). A function $f : [a, b] \to \mathbb{R}$ is said quasi-convex on $[a, b]$ if

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

for any $x, y \in [a, b]$ and $t \in [0, 1]$.

Lemma 1.2 ([5]). *Let the function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$ and $\theta, \lambda \in [0, 1]$. Then the following equality holds:*

$$\begin{aligned}
&(1-\theta)(\lambda f(a) + (1-\lambda) f(b)) + \theta f((1-\lambda)a + \lambda b) - \frac{1}{b-a}\int_a^b f(x)dx \\
= \; &(b-a)\left[-\lambda^2 \int_0^1 (t-\theta) f'(ta + (1-t)[(1-\lambda)a + \lambda b])\, dt\right. \\
&\left. + (1-\lambda)^2 \int_0^1 (t-\theta) f'(tb + (1-t)[(1-\lambda)a + \lambda b])\, dt\right].
\end{aligned}$$

In [6], İşcan gave the following theorems for quasi-convex functions.

Theorem 1.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$ and $\alpha, \lambda \in [0,1]$. If $|f'|^q$ is quasi-convex on $[a,b]$, $q > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f\left((1-\lambda)a+\lambda b\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & (b-a)\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left[\lambda^2\left(\sup\left\{|f'(a)|^q,|f'(C)|^q\right\}\right)^{\frac{1}{q}}\right. \\
&\left.+(1-\lambda)^2\left(\sup\left\{|f'(b)|^q,|f'(C)|^q\right\}\right)^{\frac{1}{q}}\right] \qquad (1.2)
\end{aligned}$$

where $C = (1-\lambda)a + \lambda b$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$ and $\alpha, \lambda \in [0,1]$. If $|f'|^q$ is quasi-convex on $[a,b]$, $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f\left((1-\lambda)a+\lambda b\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & (b-a)\left(\theta^2-\theta+\frac{1}{2}\right)\left[\lambda^2\left(\sup\left\{|f'(a)|^q,|f'(C)|^q\right\}\right)^{\frac{1}{q}}\right. \\
&\left.+(1-\lambda)^2\left(\sup\left\{|f'(b)|^q,|f'(C)|^q\right\}\right)^{\frac{1}{q}}\right] \qquad (1.3)
\end{aligned}$$

where $C = (1-\lambda)a + \lambda b$.

A refinement of Hölder integral inequality better approach than Hölder integral inequality can be given as follows:

Theorem 1.5 (Hölder-İşcan Integral Inequality [7]**).** *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then*

$$\begin{aligned}
\int_a^b |f(x)g(x)|\,dx \leq\ & \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

An refinement of power-mean integral inequality as a result of the Hölder-İşcan integral inequality can be given as follows:

Theorem 1.6 (Improved power-mean integral inequality [15]**).** *Let $q \geq 1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|$, $|g|^q$ are integrable functions on*

$[a,b]$ *then*

$$\begin{aligned}
&\int_a^b |f(x)g(x)|\,dx \\
\leq\ & \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\int_a^b (x-a)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

Our aim is to obtain the general integral inequalities giving the Hermite-Hadamard, Ostrowsky and Simpson type inequalities for the quasi-convex function in the special case using the Hölder, Hölder-İşcan integral inequalities and above lemma.

Throught this paper, we will use the following notation for shortness

$$M_1 = \left(\max\left\{|f'(A_\lambda)|^q, |f'(a)|^q\right\}\right)^{1/q} = \max\left\{|f'(A_\lambda)|, |f'(a)|\right\} \tag{1.4}$$

$$M_2 = \left(\max\left\{|f'(A_\lambda)|^q, |f'(b)|^q\right\}\right)^{1/q} = \max\left\{|f'(A_\lambda)|, |f'(b)|\right\}, \tag{1.5}$$

where $A_\lambda = (1-\lambda)\,a + \lambda b$.

2. Main results

Using Lemma 1.2 we shall give another result for quasi-convex functions as follows.

Theorem 2.1. *Let $f : I \subseteq [1,\infty)\to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$ and $\theta, \lambda \in [0,1]$. If $|f'|^q$ is quasi-convex function on the interval $[a,b]$, $q > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|(1-\theta)\left(\lambda f(a) + (1-\lambda)\,f(b)\right) + \theta f((1-\lambda)\,a + \lambda b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}2^{\frac{1}{p}}\left[\lambda^2 M_1 + (1-\lambda)^2 M_2\right]\left[N_1^{\frac{1}{p}}(\theta,p) + N_2^{\frac{1}{p}}(\theta,p)\right],
\end{aligned} \tag{2.1}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Suppose that $A_\lambda = (1-\lambda)\,a + \lambda b$. From Lemma 1.2, Hölder-İşcan integral inequality and the quasi-convexity of the function $|f'|^q$, we have

$$\begin{aligned}
&\left|(1-\theta)\left(\lambda f(a) + (1-\lambda)\,f(b)\right) + \theta f(A_\lambda) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & (b-a)\left[\lambda^2\int_0^1 |t-\theta|\,|f'(ta + (1-t)\,A_\lambda)|\,dt\right. \\
& \left.+(1-\lambda)^2\int_0^1 |t-\theta|\,|f'(tb + (1-t)\,A_\lambda)|\,dt\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq (b-a)\,\lambda^2 \left\{ \left(\int_0^1 (1-t)\,|t-\theta|^p\,dt \right)^{\frac{1}{p}} \left(\int_0^1 (1-t)\,|f'(ta+(1-t)\,A_\lambda)|^q\,dt \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_0^1 t\,|t-\theta|^p\,dt \right)^{\frac{1}{p}} \left(\int_0^1 t\,|f'(ta+(1-t)\,A_\lambda)|^q\,dt \right)^{\frac{1}{q}} \right\} \\
&\quad + (b-a)\,(1-\lambda)^2 \left\{ \left(\int_0^1 (1-t)\,|t-\theta|^p\,dt \right)^{\frac{1}{p}} \right. \\
&\quad \times \left(\int_0^1 (1-t)\,|f'(tb+(1-t)\,A_\lambda)|^q\,dt \right)^{\frac{1}{q}} \\
&\quad \left. + \left(\int_0^1 t\,|t-\theta|^p\,dt \right)^{\frac{1}{p}} \left(\int_0^1 t\,|f'(tb+(1-t)\,A_\lambda)|^q\,dt \right)^{\frac{1}{q}} \right\} \\
&\leq (b-a)\,\lambda^2 \left\{ \left[M_1 \left(\frac{1}{2}\right)^{\frac{1}{q}} N_1^{\frac{1}{p}}(\theta) + M_1 \left(\frac{1}{2}\right)^{\frac{1}{q}} N_2^{\frac{1}{p}}(\theta) \right] \right. \\
&\quad \left. + (1-\lambda)^2 \left[M_2 \left(\frac{1}{2}\right)^{\frac{1}{q}} N_1^{\frac{1}{p}}(\theta) + M_2 \left(\frac{1}{2}\right)^{\frac{1}{q}} N_2^{\frac{1}{p}}(\theta) \right] \right\} \\
&= \frac{b-a}{2} 2^{1-\frac{1}{q}} \left[\lambda^2 M_1 + (1-\lambda)^2 M_2 \right] \left[N_1^{\frac{1}{p}}(\theta) + N_2^{\frac{1}{p}}(\theta) \right] \\
&= \frac{b-a}{2} 2^{\frac{1}{p}} \left[\lambda^2 M_1 + (1-\lambda)^2 M_2 \right] \left[N_1^{\frac{1}{p}}(\theta) + N_2^{\frac{1}{p}}(\theta) \right].
\end{aligned}
$$

By simple computation

$$
\begin{aligned}
N_1(\theta,p) \quad &: \quad = \int_0^1 (1-t)\,|t-\theta|^p\,dt && (2.2) \\
&= (1-\theta)\,\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1} + \frac{\theta^{p+2}-(1-\theta)^{p+2}}{p+2} \\
N_2(\theta,p) \quad &: \quad = \int_0^1 t\,|t-\theta|^p\,dt && (2.3) \\
&= \theta\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1} + \frac{(1-\theta)^{p+2}-\theta^{p+2}}{p+2}.
\end{aligned}
$$

Thus, we obtain the inequality (2.1). This completes the proof. □

Remark 2.2. The inequality (2.1) gives better results than the inequality (1.2). Let us show that

$$
\frac{b-a}{2} 2^{\frac{1}{p}} \left[\lambda^2 M_1 + (1-\lambda)^2 M_2 \right] \left[N_1^{\frac{1}{p}}(\theta,p) + N_2^{\frac{1}{p}}(\theta,p) \right]
$$

$$
\leq (b-a) \left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1} \right)^{\frac{1}{p}}
$$

$$\times\left[\lambda^2\left(\sup\left\{|f'(a)|^q,|f'(C)|^q\right\}\right)^{\frac{1}{q}}+(1-\lambda)^2\left(\sup\left\{|f'(b)|^q,|f'(C)|^q\right\}\right)\right]^{\frac{1}{q}}.$$

Using the equalities (2.2), (2.3) and the concavity of the function $h:[0,\infty)\to\mathbb{R}$, $h(x)=x^\lambda, 0<\lambda\le 1$, by sample calculation we obtain

$$\begin{aligned}
&\frac{b-a}{2}2^{\frac{1}{p}}\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left[N_1^{\frac{1}{p}}(\theta,p)+N_2^{\frac{1}{p}}(\theta,p)\right]\\
\le\ &(b-a)2^{\frac{1}{p}}\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left[\frac{N_1(\theta,p)+N_2(\theta,p)}{2}\right]^{\frac{1}{p}}\\
=\ &(b-a)\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}},
\end{aligned}$$

which is the required.

Corollary 2.3. *Under the assumptions of Theorem 2.1 with $\theta=1$, then we have the following generalized midpoint type inequality*

$$\begin{aligned}
&\left|f((1-\lambda)a+\lambda b)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\le\frac{b-a}{2}2^{\frac{1}{p}}\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[1+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\right].
\end{aligned}\tag{2.4}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 2.4. *Under the assumptions of Theorem 2.1 with $\theta=0$, then we have the following generalized trapezoid type inequality*

$$\begin{aligned}
&\left|\lambda f(a)+(1-\lambda)f(b)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\le\frac{b-a}{2}2^{\frac{1}{p}}\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[1+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\right],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 2.5. *Under the assumptions of Theorem 2.1 with $\theta=1$, if $|f'(x)|\le M$, $x\in[a,b]$, then we have the following Ostrowski type inequality*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\\
&\le M\left(\frac{1}{2}\right)^{1-\frac{1}{p}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[1+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\right]
\end{aligned}\tag{2.5}$$

for each $x\in[a,b]$.

Proof. For each $x \in [a,b]$, there exist $\lambda_x \in [0,1]$ such that $x = (1-\lambda_x)\,a + \lambda_x b$. Hence we have $\lambda_x = \frac{x-a}{b-a}$ and $1-\lambda_x = \frac{b-x}{b-a}$. Therefore, for each $x \in [a,b]$, from the inequality (2.1) we obtain the inequality (2.5). □

Corollary 2.6. *Under the assumptions of Theorem 2.1 with $\lambda = \frac{1}{2}$ and $\theta = \frac{2}{3}$, then we have the following Simpson type inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ &\le \frac{b-a}{4}2^{\frac{1}{p}}A\left(M_1,M_2\right)\left[N_1^{\frac{1}{p}}\left(\frac{2}{3},p\right)+N_2^{\frac{1}{p}}\left(\frac{2}{3},p\right)\right],\end{aligned}$$

where A is the arithmetic mean.

Corollary 2.7. *Under the assumptions of Theorem 2.1 with $\lambda = \frac{1}{2}$ and $\theta = 1$, then we have the following midpoint type inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}2^{\frac{1}{p}}A\left(M_1,M_2\right)\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[1+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\right],$$

where A is the arithmetic mean.

Corollary 2.8. *Under the assumptions of Theorem 2.1 with $\lambda = \frac{1}{2}$ and $\theta = 0$, then we have the following trapezoid type inequality*

$$\left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}2^{\frac{1}{p}}A\left(M_1,M_2\right)\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[1+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\right],$$

where A is the arithmetic mean.

Theorem 2.9. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I such that $f' \in L\,[a,b]$, where $a,b \in I^\circ$ with $a<b$ and $\alpha,\lambda \in [0,1]$. If $|f'|^q$ is quasi-convex on $[a,b]$, $q \ge 1$, then the following inequality holds:*

$$\begin{aligned}&\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f\left((1-\lambda)a+\lambda b\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \le\;& (b-a)\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left[N_1\left(\theta\right)+N_2\left(\theta\right)\right] \qquad (2.6)\end{aligned}$$

where $C = (1-\lambda)a+\lambda b$.

Proof. Suppose that $A_{\lambda}=(1-\lambda)a+\lambda b$. From Lemma 1.2, improved power-mean integral inequality and the quasi-convexity of the function $|f'|^q$, we have

$$\begin{aligned}
&\left|(1-\theta)(\lambda f(a)+(1-\lambda)f(b))+\theta f(A_{\lambda})-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & (b-a)\left[\lambda^2\int_0^1 |t-\theta|\left|f'(ta+(1-t)A_{\lambda})\right|dt\right. \\
&\left.+(1-\lambda)^2\int_0^1 |t-\theta|\left|f'(tb+(1-t)A_{\lambda})\right|dt\right] \\
\leq\ & (b-a)\lambda^2\left\{\left(\int_0^1 (1-t)|t-\theta|dt\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\int_0^1 (1-t)|t-\theta|\left|f'(ta+(1-t)A_{\lambda})\right|^q dt\right)^{\frac{1}{q}} \\
&\left.+\left(\int_0^1 t|t-\theta|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t|t-\theta|\left|f'(ta+(1-t)A_{\lambda})\right|^q dt\right)^{\frac{1}{q}}\right\} \\
&+(b-a)(1-\lambda)^2\left\{\left(\int_0^1 (1-t)|t-\theta|dt\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\int_0^1 (1-t)|t-\theta|\left|f'(tb+(1-t)A_{\lambda})\right|^q dt\right)^{\frac{1}{q}} \\
&\left.+\left(\int_0^1 t|t-\theta|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t|t-\theta|\left|f'(tb+(1-t)A_{\lambda})\right|^q dt\right)^{\frac{1}{q}}\right\} \\
\leq\ & (b-a)\lambda^2 M_1\left[N_1(\theta)+N_2(\theta)\right]+(b-a)(1-\lambda)^2 M_2\left[N_1(\theta)+N_2(\theta)\right] \\
=\ & (b-a)\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left[N_1(\theta)+N_2(\theta)\right]
\end{aligned}$$

where

$$\begin{aligned}
N_1(\theta) &:= \int_0^1 (1-t)|t-\theta|dt=(1-\theta)\frac{\theta^2+(1-\theta)^2}{2}+\frac{\theta^3-(1-\theta)^3}{3} \\
N_2(\theta) &:= \int_0^1 t|t-\theta|dt=\theta\frac{\theta^2+(1-\theta)^2}{2}+\frac{(1-\theta)^3-\theta^3}{3}.
\end{aligned}$$

□

Remark 2.10. The inequality (2.6) coincides with the the inequality (1.3).

Using Lemma 1.2 we shall give another result for quasi convex functions as follows using the Hölder and Hölder-İşcan integral inequality. After, we will compare the results obtained with Hölder and Hölder-İşcan inequalities.

Theorem 2.11. *Let $f: I\subseteq[1,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I^{\circ}$ with $a<b$ and $\theta,\lambda\in[0,1]$. If $|f'|^q$ is quasi convex*

function on $[a,b]$, $q>1$, *then the following inequality holds:*

$$\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f((1-\lambda)a+\lambda b)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \quad (b-a)\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left(\frac{\theta^{q+1}+(1-\theta)^{q+1}}{q+1}\right)^{\frac{1}{q}} \tag{2.7}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1.2, Hölder integral inequality and quasi convexity of the function $|f'|^q$, we have

$$\begin{aligned}
&\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f(A_\lambda)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
\leq\quad & (b-a)\lambda^2\left[\int_0^1|t-\theta|\left|f'\left(ta+(1-t)A_\lambda\right)\right|dt\right.\\
&\left.+(1-\lambda)^2\int_0^1|t-\theta|\left|f'\left(tb+(1-t)A_\lambda\right)\right|dt\right]\\
\leq\quad & (b-a)\lambda^2\left(\int_0^1|t-\theta|^q\left|f'\left(ta+(1-t)A_\lambda\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+(b-a)(1-\lambda)^2\left(\int_0^1|t-\theta|^q\left|f'\left(tb+(1-t)A_\lambda\right)\right|^q dt\right)^{\frac{1}{q}}\\
\leq\quad & (b-a)\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left(\int_0^1|t-\theta|^q dt\right)^{\frac{1}{q}}\\
=\quad & (b-a)\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left(\frac{\theta^{q+1}+(1-\theta)^{q+1}}{q+1}\right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\int_0^1|t-\theta|^q dt=\frac{\theta^{q+1}+(1-\theta)^{q+1}}{q+1}. \qquad \square$$

Theorem 2.12. *Let* $f: I\subseteq[1,\infty)\to\mathbb{R}$ *be a differentiable mapping on* I° *such that* $f'\in L[a,b]$, *where* $a,b\in I^\circ$ *with* $a<b$ *and* $\theta,\lambda\in[0,1]$. *If* $|f'|^q$ *is quasi convex function on* $[a,b]$, $q>1$, *then the following inequality holds:*

$$\left|(1-\theta)\left(\lambda f(a)+(1-\lambda)f(b)\right)+\theta f((1-\lambda)a+\lambda b)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \quad (b-a)\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\lambda^2 M_1+(1-\lambda)^2 M_2\right)\left[C^{\frac{1}{q}}(\theta,q)+D^{\frac{1}{q}}(\theta,q)\right] \tag{2.8}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1.2 and by Hölder-İşcan integral inequality, we have

$$\left|(1-\theta)(\lambda f(a)+(1-\lambda)f(b))+\theta f(A_\lambda)-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{2.9}$$

$$\begin{aligned}
\leq\ & (b-a)\lambda^2\left[\int_0^1|t-\theta|\left|f'(ta+(1-t)A_\lambda)\right|dt\right.\\
& \left.+(1-\lambda)^2\int_0^1|t-\theta|\left|f'(tb+(1-t)A_\lambda)\right|dt\right]\\
\leq\ & (b-a)\lambda^2\left\{\left(\int_0^1(1-t)dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)|t-\theta|^q\left|f'(ta+(1-t)A_\lambda)\right|^q dt\right)^{\frac{1}{q}}\right.\\
& \left.+\left(\int_0^1 tdt\right)^{\frac{1}{p}}\left(\int_0^1 t|t-\theta|^q\left|f'(ta+(1-t)A_\lambda)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
& +(b-a)(1-\lambda)^2\left\{\left(\int_0^1(1-t)dt\right)^{\frac{1}{p}}\right.\\
& \times\left(\int_0^1(1-t)|t-\theta|^q\left|f'(tb+(1-t)A_\lambda)\right|^q dt\right)^{\frac{1}{q}}\\
& \left.+\left(\int_0^1 tdt\right)^{\frac{1}{p}}\left(\int_0^1 t|t-\theta|^q\left|f'(tb+(1-t)A_\lambda)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
\leq\ & (b-a)\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\lambda^2M_1+(1-\lambda)^2M_2\right)\left[C^{\frac{1}{q}}(\theta,q)+D^{\frac{1}{q}}(\theta,q)\right].
\end{aligned}$$

Since $|f'|^q$ is quasi convex function on interval $[a,b]$, the following inequalities holds.

$$\int_0^1\left|f'(ta+(1-t)A_\lambda)\right|^q dt \leq \max\left\{|f'(a)|^q,|f'(A_\lambda)|^q\right\}=M_1 \tag{2.10}$$

$$\int_0^1\left|f'(tb+(1-t)A_\lambda)\right|^q dt \leq \max\left\{|f'(b)|^q,|f'(A_\lambda)|^q\right\}=M_2 \tag{2.11}$$

Here, by simple computation we obtain

$$\int_0^1(1-t)dt=\int_0^1 tdt=\frac{1}{2},$$

$$\begin{aligned}
C(\theta,q) &= \int_0^1(1-t)|t-\theta|^q dt\\
&= (1-\theta)\left[\frac{\theta^{q+1}+(1-\theta)^{q+1}}{q+1}\right]+\left[\frac{\theta^{q+2}-(1-\theta)^{q+2}}{q+2}\right] \qquad (2.12)
\end{aligned}$$

$$\begin{aligned} D(\theta,q) &= \int_0^1 t\,|t-\theta|^q\,dt \\ &= \theta\left[\frac{\theta^{q+1}+(1-\theta)^{q+1}}{q+1}\right]-\left[\frac{\theta^{q+2}-(1-\theta)^{q+2}}{q+2}\right]. \end{aligned} \tag{2.13}$$

Thus, using (2.10)-(2.13) in (2.9), we obtain the inequality (2.8). This completes the proof. □

Remark 2.13. The inequality (2.8) is better than the inequality (2.7). For this, we need to show that

$$\begin{aligned} &(b-a)\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\lambda^2 M_1+(1-\lambda)^2 M_2\right)\left[C^{\frac{1}{q}}(\theta,q)+D^{\frac{1}{q}}(\theta,q)\right] \\ \leq\ &(b-a)\left[\lambda^2 M_1+(1-\lambda)^2 M_2\right]\left(\frac{\theta^{q+1}+(1-\theta)^{q+1}}{q+1}\right)^{\frac{1}{q}}. \end{aligned}$$

Using the inequalities (2.12), (2.13) and concavity of $\psi:[0,\infty)\to\mathbb{R}, \psi(x)=x^s, 0<s\leq 1$, we have

$$\begin{aligned} &(b-a)\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\lambda^2 M_1+(1-\lambda)^2 M_2\right)\left[C^{\frac{1}{q}}(\theta,q)+D^{\frac{1}{q}}(\theta,q)\right] \\ \leq\ &(b-a)2^{\frac{1}{q}}\left(\lambda^2 M_1+(1-\lambda)^2 M_2\right)\left(\frac{C(\theta,q)+D(\theta,q)}{2}\right)^{\frac{1}{q}} \\ =\ &(b-a)2^{\frac{1}{q}}\left(\lambda^2 M_1+(1-\lambda)^2 M_2\right)\left(\frac{1}{2}\frac{\theta^{q+1}+(1-\theta)^{q+1}}{q+1}\right)^{\frac{1}{q}} \\ =\ &(b-a)\left(\lambda^2 M_1+(1-\lambda)^2 M_2\right)\left(\frac{\theta^{q+1}+(1-\theta)^{q+1}}{q+1}\right)^{\frac{1}{q}} \end{aligned}$$

which is the required.

Corollary 2.14. *Under the assumptions of Theorem 2.12 with $\theta=1$, then we have the following generalized midpoint type inequality*

$$\begin{aligned} &\left|f((1-\lambda)a+\lambda b)-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (2.14) \\ &\leq (b-a)\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\lambda^2 M_1+(1-\lambda)^2 M_2\right)\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\left[1+\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\right]. \end{aligned}$$

where $A_\lambda=(1-\lambda)a+\lambda b$ and $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 2.15. *Under the assumptions of Theorem 2.12 with $\theta = 0$, then we have the following generalized trapezoid type inequality*

$$\left|\lambda f(a) + (1-\lambda) f(b) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq (b-a)\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\lambda^2 M_1 + (1-\lambda)^2 M_2\right)\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\left[1+\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 2.16. *Under the assumptions of Theorem 2.12 with $\theta = 1$, if $|f'(x)| \leq M$, $x \in [a,b]$, then we have the following Ostrowski type inequality*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq (b-a) M \left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\left[1+\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\right]$$

for each $x \in [a,b]$.

Proof. For each $x \in [a,b]$, there exist $\lambda_x \in [0,1]$ such that $x = (1-\lambda_x)a + \lambda_x b$. Hence we have $\lambda_x = \frac{x-a}{b-a}$ and $1-\lambda_x = \frac{b-x}{b-a}$. Therefore, for each $x \in [a,b]$, from the inequality (2.8) we obtain the desired inequality. □

Corollary 2.17. *Under the assumptions of Theorem 2.12 with $\lambda = \frac{1}{2}$ and $\theta = \frac{2}{3}$, then we have the following Simpson type inequality*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{\frac{1}{p}} A(M_1, M_2)\left[C^{\frac{1}{q}}\left(\frac{2}{3}, q\right) + D^{\frac{1}{q}}\left(\frac{2}{3}, q\right)\right]$$

where A is the arithmetic mean.

Corollary 2.18. *Under the assumptions of Theorem 2.12 with $\lambda = \frac{1}{2}$ and $\theta = 1$, then we have the following midpoint type inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{\frac{1}{p}} A(M_1, M_2)\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\left[1+\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\right],$$

where A is the arithmetic mean.

Corollary 2.19. *Under the assumptions of Theorem 2.12 with $\lambda = \frac{1}{2}$ and $\theta = 0$, then we have the following trapezoid type inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{\frac{1}{p}} A(M_1, M_2)\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\left[1+\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\right],$$

where A is the arithmetic mean.

3. Some applications for special means

Let us recall the following special means of arbitrary real numbers a, b with $a \neq b$ and $\alpha \in [0,1]$:

1. The weighted arithmetic mean

$$A_\alpha(a,b) := \alpha a + (1-\alpha)b, \quad a, b \in \mathbb{R}$$

2. The unweighted arithmetic mean

$$A(a,b) := \frac{a+b}{2}, \quad a, b \in \mathbb{R}$$

3. The weighted harmonic mean

$$H_\alpha(a,b) := \left(\frac{\alpha}{a} + \frac{1-\alpha}{b}\right)^{-1}, \quad a, b \in \mathbb{R}\setminus\{0\}$$

4. The unweighted harmonic mean

$$H(a,b) := \frac{2ab}{a+b}, \quad a, b \in \mathbb{R}\setminus\{0\}$$

5. The Logarithmic mean

$$L(a,b) := \frac{b-a}{\ln b - \ln a}, \quad a, b > 0, \ a \neq b$$

6. The n-logarithmic mean

$$L_n(a,b) := \left(\frac{b^n - a^n}{(n+1)(b-a)}\right)^{\frac{1}{n}}, \ n \in \mathbb{N}, \ a, b \in \mathbb{R}, \ a \neq b$$

Proposition 3.1. *Let $a, b \in \mathbb{R}$ with $a < b$, and $n \in \mathbb{N}, n \geq 2$. Then, for $\theta, \lambda \in [0,1]$ and $q > 1$, we have the following inequality:*

$$|(1-\theta)A_\lambda(a^n, b^n) + \theta A_\lambda^n(a,b) - L_n^n(a,b)|$$
$$\leq \frac{b-a}{2}2^{\frac{1}{p}}\left[\lambda^2 M_1 + (1-\lambda)^2 M_2\right]\left[N_1^{\frac{1}{p}}(\theta,p) + N_2^{\frac{1}{p}}(\theta,p)\right],$$

where $M_1 = \max\left\{|a|^{n-1}, |A_\lambda(a,b)|^{n-1}\right\}$, $M_2 = \max\left\{|A_\lambda(a,b)|^{n-1}, |b|^{n-1}\right\}$.

Proof. The assertion follows from the Theorem 2.1, for $f(x) = x^n, x \in \mathbb{R}$. □

Proposition 3.2. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, and $\theta, \lambda \in [0,1]$. Then, for $q > 1$, we have the following inequality:*

$$\begin{aligned}
&\left|(1-\theta) H_{\lambda}^{-1}(a,b) + \theta A_{\lambda}^{-1}(a,b) - L^{-1}(a,b)\right| \\
&\leq \frac{b-a}{2} 2^{\frac{1}{p}} \left[\lambda^2 M_1 + (1-\lambda)^2 M_2\right] \left[N_1^{\frac{1}{p}}(\theta,p) + N_2^{\frac{1}{p}}(\theta,p)\right],
\end{aligned}$$

where $M_1 = \max\left\{a^{-2}, A_{\lambda}^{-2}(a,b)\right\}$, $M_2 = \max\left\{A_{\lambda}^{-2}(a,b), b^{-2}\right\}$.

Proof. The assertion follows from the Theorem 2.1, for $f(x) = \frac{1}{x}, x \in (0,\infty)$. □

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Generalized fractional integral operator in a complex domain

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Abstract. A new fractional integral operator is used to present a generalized class of analytic functions in a complex domain. The method of definition is based on a Hadamard product of analytic function, which is called convolution product. Then we formulate a convolution integral operator acting on the subclass of normalized analytic functions. Consequently, we investigate the suggested convolution operator geometrically. Differential subordination inequalities, taking the starlike formula are given. Some consequences of well known results are illustrated.

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1. Introduction

Many scholars, academic and researchers have applied fractional order integral operators (FOIOs) in real-world situations in a variety of scientific and technological sectors in recent years. It is well known, there are a number of definitions of FOIOs that can be utilized to solve fractional integral equations employing special functions (SFs). Fractional differentiation and integration using the extended Mittag-Leffler kernel were proposed in 2016 [1] and drew interest from a wide range of research sectors. Many features of these differential and integral operators have been noticed in real-world applications, such as crossover behavior (see [28]). These classes of specialized

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functions [8, 15] have newly become crucial in the fields of almost all applied sciences [20], natural science, engineering and computer science (see [2], [9], [24],[25]).

Integrals and the outputs of many different forms of differential equations are examples of special functions. As a consequence, record integrals involve explanations of SFs, which take account of the furthermost fundamental integrals; at the actual slightest, the integral representation of SFs. Because differential operators are important in mathematical sciences and applied mathematics, the theory of SFs is tightly linked to various physics topics [7].

In this note, we investigated the features of the k-Raina function under FOIOs and created several novel images. Via their extended character and utility in the theory of integral operators and a crucial part of computational mathematics, the conclusions produced here involve special classes of analytic functions such as the k-Mittag-Leffler function, S-function and K-function. Our methodology is based on the theory of differential subordination to present a set of differential inequalities type starlikeness in a complex domain.

2. Techniques

Here, we'll proceed over the methods we utilized.

2.1. Geometric approaches

The following concepts can be found in [16]

Definition 2.1. The set $\mathbb{O} := \{\chi \in \mathbb{C} : |\chi| < 1\}$, is the open unit disk in $z-$plane. The analytic functions Σ_1, Σ_2 in $\mathbb{O}$ are under the subordinated inequality $\Sigma_1 \prec \Sigma_2$ or

$$\Sigma_1(\chi) \prec \Sigma_2(\chi), \quad \chi \in \mathbb{O}$$

if for an analytic function $\varsigma, |\varsigma| \le |\chi| < 1$ holds such that $\Sigma_1(\chi) = \Sigma_2(\varsigma(\chi)), \chi \in \mathbb{O}$.

Definition 2.2. The class of all regular functions given by

$$\sigma(\chi) = \chi + \sum_{n=2}^{\infty} a_n\, \chi^n, \quad \chi \in \mathbb{O}, \quad \sigma(0) = \sigma'(0) - 1 = 0,$$

is denoted by $\aleph$. Moreover, the analytic functions $\sigma_1, \sigma_2 \in \aleph$ are convoluted ($\sigma_1 * \sigma_2$) if they have the Hadamard product [22]

$$(\sigma_1 * \sigma_2)(\chi) = \left(\chi + \sum_{n=2}^{\infty} a_n\, \chi^n\right) * \left(\chi + \sum_{n=2}^{\infty} b_n\, \chi^n\right) = \chi + \sum_{n=2}^{\infty} a_n\, b_n\, \chi^n.$$

Definition 2.3. Define the following class of regular functions

$$\mathcal{P} := \{p : p(\chi) = 1 + \ell_1\chi + \ell_2\chi^2 + ..., \chi \in \mathbb{O}, \Re(p(\chi)) > 0, p(0) = 1\}.$$

Special sub-classes of $\mathcal{P}$ are the starlike subclass of functions $\sigma \in \aleph$ satisfying the functional

$$S_\sigma(\chi) = \frac{\chi\sigma'(\chi)}{\sigma(\chi)};$$

and the convex subclass of functions $\sigma \in \aleph$ having the functional

$$K_\sigma(\chi) = 1 + \frac{\chi\sigma''(\chi)}{\sigma'(\chi)}.$$

2.2. Raina's function

Let's start with the Raina's function (RAF), which is a familiar special feature.

Definition 2.4. In [21], the definition of RAF

$$\rho^{\mu}_{\alpha,\beta}(\chi) = \sum_{n=0}^{\infty} \frac{\mu(n)}{\Gamma(\alpha n + \beta)} \chi^n, \quad \chi \in \mathbb{O}.$$

$$\Big(\alpha, \quad \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \quad \mu := \{\mu(0), \mu(1), ..., \mu(n)\}, \mu(j) \in \mathbb{C} \forall j = 0, ..., n\Big)$$

Remark 2.5.

- $n \geq 0, \mu(n) = 1 \Rightarrow \rho_{\alpha,\beta}(\chi) = \sum_{n=0}^{\infty} \frac{\chi^n}{\Gamma(\alpha n + \beta)}$, the Mittag-Leffler function.
- $\alpha = \beta = 1, \mu(n) = \frac{(v)_n (w)_n}{(u)_n} \Rightarrow {}_2G_1(v, w; u; \chi) = \sum_{n=0}^{\infty} \frac{(v)_n (w)_n}{(u)_n} \frac{\chi^n}{\Gamma(n+1)}$,
  the hypergeometric function.
- $\mu(n) = \frac{1}{n!} \frac{(w_1)_n ... (w_{k_1})_n}{(u_1)_n ... (u_{k_2})_n} \Rightarrow \mathbb{M}^{k_1,k_2}_{\alpha,\beta}(\chi) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} \frac{(w_1)_n ... (w_{k_1})_n}{(u_1)_n ... (u_{k_2})_n} \frac{\chi^n}{n!}$
  the $\mathbb{M}-$series [27].
- $\mu(n) = \frac{(v)_n}{n!} \frac{(w_1)_n ... (w_{k_1})_n}{(u_1)_n ... (u_{k_2})_n} \Rightarrow \mathbb{K}^{k_1,k_2,v}_{\alpha,\beta}(\chi) = \sum_{n=0}^{\infty} \frac{(v)_n}{\Gamma(\alpha n + \beta)} \frac{(w_1)_n ... (w_{k_1})_n}{(u_1)_n ... (u_{k_2})_n} \frac{\chi^n}{n!}$
  the $\mathbb{K}-$function [26].

2.3. Complex Raina's FOIOs

The Raina's FOIO is defined for analytic function $f(z), z \in \mathbb{C}$ in a complex domain containing the origin ($\mathbb{O}$) by the formula

$$I^{\mu,\tau}_{\alpha,\beta} f(\chi) = \int_0^{\chi} (\chi - z)^{\beta-1} \rho^{\mu}_{\alpha,\beta}[\tau(\chi - z)^{\alpha}] f(z) dz \tag{2.1}$$

$$\Big(\Re(\alpha) > 0, \quad \Re(\beta) > 0, \quad \chi, \ z \in \mathbb{C}, \quad \tau \in \mathbb{R}\Big).$$

Note that the integral $I^{\mu,\tau}_{\alpha,\beta} f(\chi)$ involves the well known Riemann-Liouville integral operator, when $\tau = 0$ and $\mu(0) = 1$

$$I_\beta f(\chi) = \frac{1}{\Gamma(\beta)} \int_0^{\chi} (\chi - z)^{\beta-1} f(z) dz \quad \Re(\beta) > 0$$

whenever the function $f(\chi)$ is analytic in simply-connected region of the complex z-plane.

In general, we have the integrals

$$\int_0^\chi (\chi - z)^{\beta-1}\rho^\mu_{\alpha,\beta}[\tau(\chi - z)^\alpha]dz = \chi^\beta \rho^\mu_{\alpha,\beta+1}[\tau(\chi - z)^\alpha]$$

$$\int_0^\chi (\chi - z)^{\beta}\rho^\mu_{\alpha,\beta+1}[\tau(\chi - z)^\alpha]dz = \chi^{\beta+1} \rho^\mu_{\alpha,\beta+2}[\tau(\chi - z)^\alpha]$$

$$\vdots$$

$$\begin{aligned}\int_0^\chi (\chi - z)^{\beta+m}\rho^\mu_{\alpha,\beta+1+m}[\tau(\chi - z)^\alpha]dz &= \chi^{\beta+m+1} \rho^\mu_{\alpha,\beta+m+2}[\tau(\chi - z)^\alpha]\\ &= \chi^{\beta+m+1}\sum_{n=0}^\infty \frac{\mu(n)}{\Gamma(\alpha n+\beta+m+2)}\,[\tau(\chi - z)^\alpha]^n.\end{aligned}$$

Moreover, we have the integral

$$\begin{aligned}I^{\mu,\tau}_{\alpha,\beta,m}(\chi) &:= \int_0^1 (\chi)^{\beta+m+\alpha-1}\rho^\mu_{\alpha,\beta+1+m}[\tau(\chi)^\alpha\chi^{1-\alpha}]d\chi\\ &= \chi^{\alpha+\beta+m}\sum_{n=0}^\infty \frac{\tau^n\mu(n)}{\Gamma(\alpha n+\beta+m+2)}\,\chi^n.\end{aligned}$$

To normalize the above integral, we define the functional integral formula, as follows:

$$\begin{aligned}\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}(\chi) &:= \left(\frac{\Gamma(\alpha+\beta+m+2)}{\tau\mu(1)}\right)\left(\frac{I^{\mu,\tau}_{\alpha,\beta,m}(\chi)}{\chi^{\alpha+\beta+m}} - \frac{\mu(0)}{\Gamma(\beta+m+2)}\right) \qquad (2.2)\\ &= \sum_{n=0}^\infty \left(\frac{\Gamma(\alpha+\beta+m+2)}{\tau\mu(1)}\right)\left(\frac{\tau^n\mu(n)}{\Gamma(\alpha n+\beta+m+2)}\right)\chi^n\\ &= \chi + \sum_{n=2}^\infty \left(\frac{\Gamma(\alpha+\beta+m+2)}{\tau\mu(1)}\right)\left(\frac{\tau^n\mu(n)}{\Gamma(\alpha n+\beta+m+2)}\right)\chi^n,\end{aligned}$$

where $\tau \neq 0$, $\mu(1) \neq 0$, $m \in \mathbb{Z}$. It is clear that $\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}(\chi) \in \aleph$.

2.4. Convoluted fractional operator

We continue to define the convolution operator using the Hadamard product combining the suggested integral $\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}(\chi)$ with the function $\sigma \in \aleph$. The main integral convoluted operator in this effort is given, as follows:

$$\begin{aligned}&\left(\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma\right)(\chi) \qquad (2.3)\\ &= \left(\chi + \sum_{n=2}^\infty \left(\frac{\Gamma(\alpha+\beta+m+2)}{\tau\mu(1)}\right)\left(\frac{\tau^n\mu(n)}{\Gamma(\alpha n+\beta+m+2)}\right)\chi^n\right) * \left(\chi + \sum_{n=2}^\infty a_n\chi^n\right)\\ &= \chi + \sum_{n=2}^\infty \left(\frac{\Gamma(\alpha+\beta+m+2)\tau^{n-1}\mu(n)}{\mu(1)\Gamma(\alpha n+\beta+m+2)}\right) a_n\chi^n.\end{aligned}$$

Obviously, the convolution integral operator $\left(\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma\right) \in \aleph$.

Remark 2.6. By assuming the factor $\mu(n)$ for any coefficient formulas, we obtain all the convoluted operators, differential operators (like the Sàlàgean differential operator [23] and its generalizations [3]), fractional differential operators (Caputo and its generalizations, ABC-differential operator), integral operators (like the Sàlàgean integral operator [23]), fractional differential and integral operators [11], symmetric differential and integral operators [10], mixed fractional operators in the open unit disk [13], convoluted operator (such as the Carlson and Shaffer convoluted operator [5], Ruscheweyh convoluted operator [22], Noor operator [17] and Attiya operator [4]), special series (like $\mathbb{M}-$series, $\mathbb{S}-$series [29] and the Borel distribution [30]), mixed differential operators and all special functions in the litterateurs including the quantum calculus [12, 18].

Example 2.7.

- $\tau = 1, m = -2, \mu(n) = \dfrac{1}{n!}\dfrac{\Gamma(\gamma + n\kappa)}{\Gamma(\gamma+\kappa)}$, $\forall n \geq 1$, and $\gamma \in \mathbb{C}$ and $\Re(\kappa) > 0$ then we obtain the convoluted operator in [4]

$$\left(\mathbb{I}^{\mu,1}_{\alpha,\beta,-2} * \sigma\right)(\chi) = \chi + \sum_{n=2}^{\infty}\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha n+\beta)}\frac{\Gamma(\gamma+n\kappa)}{\Gamma(\gamma+\kappa)}\frac{1}{n!}\right) a_n \chi^n.$$

As special cases from the above series, when $\alpha = 0, \gamma = \kappa = 1$, we have $\left(\mathbb{I}^{\mu,1}_{0,\beta,-2} * \sigma\right)(\chi) = \sigma(\chi)$. And for $\alpha = 0$, $\gamma = 2$, $\kappa = 1$, we obtain the operator

$$\left(\mathbb{I}^{\mu,1}_{0,\beta,-2} * \sigma\right)(\chi) = \frac{1}{2}[\sigma(\chi) + \chi\sigma'(\chi)].$$

Moreover, when $\alpha = \gamma = \kappa = 1, \beta = 0, \sigma(\chi) = \dfrac{\chi}{1-\chi}$, we have

$$\left(\mathbb{I}^{\mu,1}_{1,0,-2} * \sigma\right)(\chi) = \chi\, e^{\chi}.$$

Finally, when $\alpha = \gamma = \kappa = 1, \beta = 1, \sigma(\chi) = \dfrac{\chi}{1-\chi}$, we get

$$\left(\mathbb{I}^{\mu,1}_{1,1,-2} * \sigma\right)(\chi) = e^{\chi} - 1.$$

- The Operator (2.3) satisfies the recurrent relation

$$\alpha\chi[\left(\mathbb{I}^{\mu,\tau}_{\alpha,\beta+1,m} * \sigma\right)(\chi)]' = (\alpha+\beta)\left(\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma\right)(\chi) - \beta\left(\mathbb{I}^{\mu,\tau}_{\alpha,\beta+1,m} * \sigma\right)(\chi).$$

Note that, when $\tau = 1, m = -2, \mu(n) = \dfrac{1}{n!}\dfrac{\Gamma(\gamma+n\kappa)}{\Gamma(\gamma+\kappa)}$, $\forall n \geq 1$, and $\gamma \in \mathbb{C}$ and $\Re(\kappa) > 0$ then we have [4]-Lemma 2.1. And under the same set of parameters, with $\sigma(\chi) = \dfrac{\chi}{1-\chi}$, we obtain the result in [28]-Theorem 2.1. Finally, if $\alpha = 1$, we have the equation (1.8) in [6].

In the following section, we illustrate our results concerning the generalized Raina FOIO of a complex variable.

3. Results

In this place, we discuss the sufficient conditions of the Ma-Minda starlike inequality [14]

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)=\frac{\chi\left(\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma\right)'(\chi)}{\left(\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma\right)(\chi)}\prec\Lambda(\chi).$$

For this purpose, we request the next result [16] (Corollary 3.4h.1 p.135).

Lemma 3.1. *Suppose that λ is analytic and Λ is univalent in $\mathbb{O}$ with $\lambda(0)=\Lambda(0)$, and an analytic function ℓ defined in a domain involving $\Lambda(\mathbb{O})$ and $\Lambda(\mathbb{O})$. If $\chi\Lambda'(\chi)\ell(\Lambda(\chi))$ is starlike, then the relation*

$$\chi\lambda'(\chi)\ell(\lambda(\xi))\prec\chi\Lambda'(\chi)\ell(\Lambda(\chi))$$

yields $\lambda(\chi)\prec\Lambda(\chi)$ and Λ is the best dominant.

Theorem 3.2. *Take into consideration the following hypotheses:*

(i) $\sigma\in\aleph$, Λ *is univalent in* $\mathbb{O}$;

(ii) $\dfrac{\chi\Lambda'(\chi)}{\Lambda(\chi)(\Lambda(\chi)-1)}$ *is starlike in* $\mathbb{O}$;

(iii) $\dfrac{K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-1}{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-1}\prec 1+\dfrac{\chi\Lambda'(\chi)}{\Lambda(\chi)(\Lambda(\chi)-1)}$ *occurs.*

Then

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\prec\Lambda(\chi),\quad \chi\in\mathbb{O}$$

and Λ is the best dominant.

Proof. Denotes Ω, as follows:

$$\Omega(\chi):=S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi),\quad \chi\in\mathbb{O}.$$

Thus, a computation implies

$$S_\Omega(\chi)=K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-\Omega(\chi).$$

Substituting implies that

$$\begin{aligned}\frac{K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-1}{T_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-1}&=\frac{S_\Omega(\chi)+\Omega(\chi)-1}{\Omega(\chi)-1}\\&=1+\frac{\chi\Omega'(\chi)}{\Omega(\chi)(\Omega(\chi)-1)}.\end{aligned}$$

Consequently, we obtain

$$\frac{\chi\Omega'(\chi)}{\Omega(\chi)(\Omega(\chi)-1)}\prec\frac{\chi\Lambda'(\xi)}{\Lambda(\chi)(\Lambda(\chi)-1)},\quad \chi\in\mathbb{O}.$$

In view of Lemma 3.1, we attain the result. □

Theorem 3.3. *Assume the following hypotheses*

(i) $\sigma\in\aleph,\Lambda$ *is univalent in* $\mathbb{O}$;

(ii) $\dfrac{\chi\Lambda'(\chi)}{\Lambda(\chi)-1}$ *is starlike in* $\mathbb{O}$;

(iii) $S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\left(\dfrac{K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\xi)-p}{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-1}-1\right)\prec\dfrac{\chi\Lambda'(\chi)}{\Lambda(\chi)-1}$ *holds.*

Then

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\prec\Lambda(\chi),\quad \chi\in\mathbb{O}$$

and Λ *is the best dominant.*

Proof. Consider the function Ω, as follows:

$$\Omega(\chi):=S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi),\quad \chi\in\mathbb{O}.$$

Accordingly, we have

$$S_\Omega(\chi)+\Omega(\chi)=K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi).$$

A calculation yields

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\left(\frac{K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-1}{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-1}-1\right)=\frac{\chi\Omega'(\chi)}{\Omega(\chi)-1}.$$

Which leads to

$$\frac{\chi\Omega'(\chi)}{\Omega(\chi)-1}\prec\frac{\chi\Lambda'(\chi)}{\Lambda(\chi)-1},\quad \chi\in\mathbb{O}.$$

In virtue of Lemma 3.1, we get the desired outcome. □

Theorem 3.4. *Consider the following assumptions*

(i) $\sigma\in\aleph,\Lambda$ *is univalent in* $\mathbb{O}$;

(ii) S_Λ *IS starlike in* $\mathbb{O}$;

(iii) $K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\prec S_\Lambda(\chi)$ *satisfies.*

Then

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\prec\Lambda(\chi),\quad \chi\in\mathbb{O}$$

and Λ *is the best dominant.*

Proof. Let Ω as follows:

$$\Omega(\chi):=S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\xi),\quad \chi\in\mathbb{O}.$$

Thus, we get

$$S_\Omega(\chi)+\Omega(\chi)=K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi).$$

Consequently, we have

$$K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)=S_\Omega(\chi).$$

Hence,

$$S_\Omega(\chi)\prec S_\Lambda(\chi),\quad \chi\in\mathbb{O}.$$

Finally, Lemma 3.1 yields the outcome $\Omega(\chi)\prec\Lambda(\chi)$. □

Theorem 3.5. *Use these assumptions:*

(i) $\sigma\in\aleph$, Λ *is univalent in* $\mathbb{O}$;

(ii) $\chi\Lambda'(\chi)$ *is starlike in* $\mathbb{O}$;

(iii) $S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\left(K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\right)\prec\chi\Lambda'(\chi)$ *occurs.*
Then

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\prec\Lambda(\chi),\quad\chi\in\mathbb{O}$$

and Λ *is the best dominant.*

Proof. Formulate the function Ω by:

$$\Omega(\chi):=S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi),\quad\chi\in\mathbb{O}.$$

Thus, we have

$$S_\Omega(\chi)+\Omega(\chi)=K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi).$$

Substituting attains

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\left(K_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)-S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\right)=\chi\Omega'(\chi).$$

Hence,

$$\chi\Omega'(\chi)\prec\chi\Lambda'(\chi),\quad\chi\in\mathbb{O}.$$

By Lemma 3.1, we obtain $\Omega(\chi)\prec\Lambda(\chi)$. □

Theorem 3.6. *Suppose that* Λ *is convex univalent in* $\mathbb{O}$ *satisfying the inequality*

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)\prec\Lambda(\chi),\quad\chi\in\mathbb{O},$$

where $\Lambda(0)=1$. *Then*

$$[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]\prec\chi\exp\left(\int_0^\chi\frac{\Lambda(w(\xi))}{\xi}d\xi\right),$$

where w *has the properties* $w(0)=0$ *and* $|w(\chi)|<1$. *In addition, the inequality* $|\chi|:=\rho<1$ *yields*

$$\exp\left(\int_0^1\frac{\Lambda(-\rho)}{\rho}d\rho\right)\le\left|\frac{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}{\chi}\right|\le\exp\left(\int_0^1\frac{\Lambda(\rho)}{\rho}d\rho\right).$$

Proof. A computation implies

$$\frac{\left([\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)\right)'}{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}-\frac{1}{\chi}=\frac{\Lambda(w(\chi))-1}{\chi}.$$

Integration yields

$$[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)\prec\chi\exp\left(\int_0^\chi\frac{\Lambda(w(\xi))}{\xi}d\xi\right),$$

which leads to

$$\frac{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}{\chi}\prec\exp\left(\int_0^\chi\frac{\Lambda(w(\xi))}{\xi}d\xi\right).$$

But,

$$\Lambda(-\rho|\chi|)\le\Re\left(\Lambda(w(\chi\rho))\right)\le\Lambda(\rho|\chi|)$$

then, we obtain

$$\int_0^1\frac{\Lambda(-\rho|\chi|)}{\rho}d\rho\le\int_0^1\frac{\Re\left(\Lambda(w(\chi\rho))\right)}{\rho}d\rho\le\int_0^1\frac{\Lambda(\rho|\chi|)}{\rho}d\rho.$$

A combination of the last two relations, we attain

$$\int_0^1 \frac{\Lambda(-\rho|\chi|)}{\rho}d\rho \leq \log\Big|\frac{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma](\chi)}{\chi}\Big| \leq \int_0^1 \frac{\Lambda(\rho|\chi|)}{\rho}d\rho.$$

Which imposes

$$\exp\left(\int_0^1 \frac{\Lambda(-\rho)}{\rho}d\rho\right) \leq \Big|\frac{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma](\chi)}{\chi}\Big| \leq \exp\left(\int_0^1 \frac{\Lambda(\rho)}{\rho}d\rho\right). \qquad \square$$

Corollary 3.7. *In Theorem 3.6, let*

$$\mu(n) = \frac{1}{n!}\frac{\Gamma(\gamma+n\kappa)}{\Gamma(\gamma+\kappa)},$$

with $\alpha = 0, \gamma = \kappa = 1, m = -2 \Rightarrow \left(\mathbb{I}^{\mu,1}_{0,\beta,-2} * \sigma\right)(\chi) = \sigma(\chi)$.
Consider that Λ *is convex univalent in* $\mathbb{O}$ *with*

$$S_\sigma(\chi) \prec \Lambda(\chi), \quad \xi \in \mathbb{O},$$

where $\Lambda(0) = 1$ *then*

$$\sigma(\chi) \prec \chi \exp\left(\int_0^\chi \frac{\Lambda(w(\xi))}{\xi}d\xi\right),$$

where w *as in Theorem 3.6. In addition, the relation* $|\chi| := \rho < 1$ *gives*

$$\exp\left(\int_0^1 \frac{\Lambda(-\rho)}{\rho}d\rho\right) \leq \Big|\frac{\sigma(\chi)}{\chi}\Big| \leq \exp\left(\int_0^1 \frac{\Lambda(\rho)}{\rho}d\rho\right).$$

Lastly, we present a special result when $\Lambda(\chi) := \dfrac{1+\phi\chi}{1+\psi\chi}$, where $-1 \leq \psi < \phi \leq 1$.

Theorem 3.8. *Consider the generalized FOIO* (2.3).

(i) *If the following subordination holds:*

$$\left(S_{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)}(\chi) - 1\right)[S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}]^{-1} + 1 \prec \frac{(\phi-\psi)\chi(2+\phi\chi)}{(1+\phi\chi)^2},$$

then

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)} \prec \frac{1+\phi\chi}{1+\psi\chi}.$$

$$\Big(-1 \leq \psi < \phi \leq 0, \quad \chi \in \mathbb{O}\Big)$$

Moreover,

$$\frac{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma](\chi)}{\chi} \prec (1+\psi\chi)^{\frac{\phi-\psi}{\psi}}, \quad \psi \neq 0.$$

(ii) *If the next inequality occurs*

$$S_{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)}(\chi) + 1 \prec \frac{1+\phi\chi}{1+\psi\chi}$$

then

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)} \prec (1+\psi\chi)^{\frac{\phi-\psi}{\psi}}.$$

$$\Big(|\frac{\phi-\psi}{\psi} \pm 1| \le 1, \quad \chi \in \mathbb{O}\Big)$$

(iii) *If the following relation exists*

$$S_{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)}(\chi) + 1 \prec 1 + \phi\chi$$

then

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)} \prec e^{\phi\chi}.$$

$$\Big(\psi = 0, \quad |\phi| < \pi, \quad \chi \in \mathbb{O}\Big)$$

In addition,

$$\frac{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}{\chi} \prec e^{\phi\chi}.$$

All the above results are the best dominant.

Proof. Clearly, based on the definition of the functional S, we have

$$S_{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)}(\chi) = \frac{\chi\Big(S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}\Big)'}{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}},$$

where $S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(0) = 1$. Now let

$$\Omega(\chi) := S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi).$$

Then a calculation implies

$$\begin{aligned} 1 - \frac{1}{\Omega(\chi)} + \frac{\chi\Omega'(\chi)}{\Omega^2(\chi)} &= 1 - \frac{1}{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)} + \frac{\chi[S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)]'}{[S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)]^2} \\ &= \Big(S_{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)}(\chi) - 1\Big)[S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}]^{-1} + 1 \\ &\prec \frac{(\phi-\psi)\chi(2+\phi\chi)}{(1+\phi\chi)^2}. \end{aligned}$$

Then in view of [19]-Lemma 3, we have the outcome in (i). Since we have

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)} \prec \frac{1+\phi\chi}{1+\psi\chi}.$$

$$\Big(-1 \le \psi < \phi \le 0, \quad \chi \in \mathbb{O}\Big)$$

Then the second inequality comes from [19]-Theorem 2

$$\frac{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)}{\chi} \prec (1+\psi\chi)^{\frac{\phi-\psi}{\psi}}, \quad \psi \neq 0.$$

We aim to prove (ii). Since

$$\begin{aligned} 1 + \frac{\chi\Omega'(\chi)}{\Omega(\chi)} &= 1 + \frac{\chi[S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)]'}{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)} \\ &= S_{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)}(\chi) + 1 \\ &\prec \frac{1+\phi\chi}{1+\psi\chi}. \end{aligned}$$

Then in view of [19]-Lemma 4(i), we obtain the result in (ii).
A computation yields

$$\begin{aligned} 1 + \frac{\chi\Omega'(\chi)}{\Omega(\chi)} &= 1 + \frac{\chi[S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)]'}{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)} \\ &= S_{S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma]}(\chi)}(\chi) + 1 \\ &\prec 1+\phi\chi. \end{aligned}$$

Then in view of [19]-Lemma 4(ii), we get the result in (iii). Since

$$S_{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m}*\sigma](\chi)} \prec e^{\phi\chi},$$

$$\Big(\psi = 0, \quad |\phi| < \pi, \quad \chi \in \mathbb{O}\Big)$$

we attain the second part by using [19]-Theorem 2

$$\frac{[\mathbb{I}^{\mu,\tau}_{\alpha,\beta,m} * \sigma](\chi)}{\chi} \prec e^{\phi\chi}. \qquad \square$$

Corollary 3.9. [19] *In Theorem 3.8, let*

$$\mu(n) = \frac{1}{n!}\frac{\Gamma(\gamma+n\kappa)}{\Gamma(\gamma+\kappa)},$$

with $\alpha = 0, \gamma = \kappa = 1, m = -2 \Rightarrow \Big(\mathbb{I}^{\mu,1}_{0,\beta,-2} * \sigma\Big)(\chi) = \sigma(\chi).$

(i) *If the following subordination holds:*

$$\big(S_{S_{\sigma(\chi)}}(\chi) - 1\big)\,[S_{\sigma(\chi)}]^{-1} + 1 \prec \frac{(\phi-\psi)\chi(2+\phi\chi)}{(1+\phi\chi)^2},$$

then

$$S_{\sigma(\chi)} \prec \frac{1+\phi\chi}{1+\psi\chi}.$$

$$\Big(-1 \le \psi < \phi \le 0, \quad \chi \in \mathbb{O}\Big)$$

Moreover,

$$\frac{\sigma(\chi)}{\chi} \prec (1+\psi\chi)^{\frac{\phi-\psi}{\psi}}, \quad \psi \neq 0.$$

(ii) *If the next inequality occurs*

$$S_{S_{\sigma(\chi)}}(\chi) + 1 \prec \frac{1+\phi\chi}{1+\psi\chi}$$

then

$$S_{\sigma(\chi)} \prec (1+\psi\chi)^{\frac{\phi-\psi}{\psi}}.$$

$$\Big(|\frac{\phi-\psi}{\psi} \pm 1| \le 1, \quad \chi \in \mathbb{O}\Big)$$

(iii) *If the following relation exists*

$$S_{S_{\sigma(\chi)}}(\chi) + 1 \prec 1 + \phi\chi$$

then

$$S_{\sigma(\chi)} \prec e^{\phi\chi}.$$

$$\Big(\psi = 0, \quad |\phi| < \pi, \quad \chi \in \mathbb{O}\Big)$$

In addition,

$$\frac{\sigma(\chi)}{\chi} \prec e^{\phi\chi}.$$

All the above results are the best dominant.

Example 3.10. Under the assumptions of Corollary 3.9, we have the following examples (see Fig. 1):

- For $\phi = 1 - 2a, a \in [0,1), \psi = -1$, we get

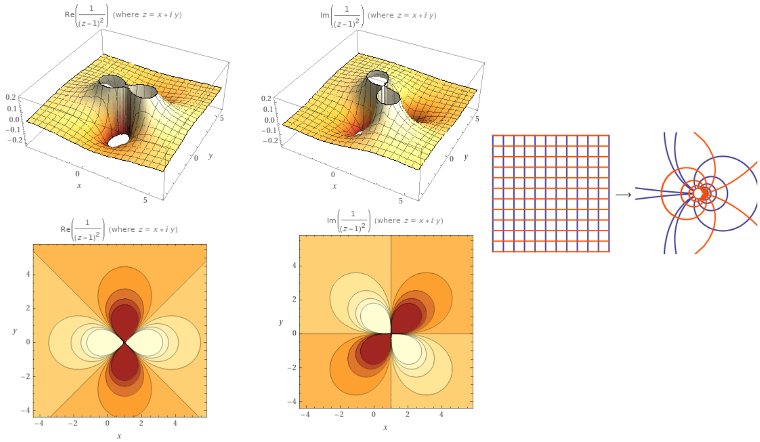
$$\frac{\sigma(\chi)}{\chi} \prec \frac{1}{(1-\chi)^{2(1-a)}}.$$

- For $a = 0$, we obtain

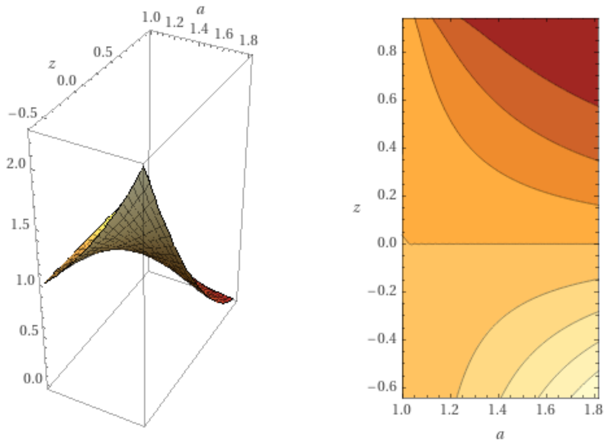
$$\frac{\sigma(\chi)}{\chi} \prec \frac{1}{(1-\chi)^{2}}.$$

4. Conclusion

The Generalized fractional integral operator is formulated using the Raina's function. The suggested FOIO is a generalization of many operators and series. We formulated the FOIO in classes of starlike functions and explored the sufficient conditions for these classes. Many recent results are conformed as special cases. We suggest to include it in different other classes of analytic functions, for the next step of research.



(A) Plotting of $\dfrac{1}{(1-\chi)^2}, a = 0$



(B) Plotting of $\dfrac{1}{(1-\chi)^{2(1-a)}}, a \neq 0$

FIGURE 1. Functions in Example 3.10

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Symmetric Toeplitz determinants for classes defined by post quantum operators subordinated to the limaçon function

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Thirumalai Vinjimur Sudharsan and Teodor Bulboacă

Dedicated to the memory of Professor Bhaskaran Adolf Stephen (1966–2017)

Abstract. The present extensive study is focused to find estimates for the upper bounds of the Toeplitz determinants. The logarithmic coefficients of univalent functions play an important role in different estimates in the theory of univalent functions, and in the this paper we derive the estimates of Toeplitz determinants and Toeplitz determinants of the logarithmic coefficients for the subclasses $\mathrm{L}_s\mathcal{S}_p^q$, $\mathrm{L}_s\mathcal{C}_p^q$, and $\mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{S}$, $\mathrm{L}_s\mathcal{C}_p^q \cap \mathcal{S}$, $0 < q \leq p \leq 1$, respectively, defined by post quantum operators, which map the open unit disc $\mathbb{D}$ onto the domain bounded by the limaçon curve defined by $\partial\mathcal{D}_s := \left\{u+iv \in \mathbb{C} : \left[(u-1)^2+v^2-s^4\right]^2 = 4s^2\left[(u-1+s^2)^2+v^2\right]\right\}$, where $s \in [-1,1] \setminus \{0\}$.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. If $f \in \mathcal{A}$, then

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in \mathbb{D}, \tag{1.1}$$

and denotes by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$ (see [6] for details).

For two functions f and g analytic in $\mathbb{D}$, we say that the function f is *subordinate* to g in $\mathbb{D}$, and write $f(z) \prec g(z)$, if there exists an analytic function in $\mathbb{D}$ denoted by w, with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{D}$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In particular, if the function g is univalent in $\mathbb{D}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

We recall that $\mathcal{B}$ denote the class of analytic self-mappings of the unit disc, that maps the origin onto the origin [13], that is

$$\mathcal{B} := \left\{ w(z) = \sum_{n=1}^{\infty} w_n z^n : |w(z)| < 1, \ z \in \mathbb{D} \right\}, \tag{1.2}$$

and the class $\mathcal{B}$ is known as the class of *Schwarz functions*.

In 2018, Yunus et. al. [21] studied the subclass of starlike functions associated with a limaçon domain. The limaçon of Pascal also known as limaçon is a curve that in polar coordinates has the form $r = b + a\cos\theta$, where a and b are real positive real and $\theta \in (0, 2\pi)$. If $b \geq 2a$ the limaçon is a convex curve and if $2a > b > a$ it has an indentation bounded by two inflection points. For $b = a$ the limaçon degenerates to a cardioid.

Recently, Kanas et. al. [13] introduced subclasses $ST_L(s)$ and $CV_L(s)$ of starlike and convex function respectively. Geometrically, they consist of functions $f \in \mathcal{A}$ such that $\dfrac{zf'(z)}{f(z)}$ and $\dfrac{\left(zf'(z)\right)'}{f'(z)}$ lie in the region bounded by the limaçon curve defined as

$$\partial\mathcal{D}_s := \left\{ u + iv \in \mathbb{C} : \left[(u-1)^2 + v^2 - s^4\right]^2 = 4s^2 \left[\left(u - 1 + s^2\right)^2 + v^2\right] \right\},$$

where $s \in [-1, 1] \setminus \{0\}$. If we define the *limaçon function*

$$\mathbb{L}_s(z) := (1 + sz)^2, \ s \in [-1, 1] \setminus \{0\}, \tag{1.3}$$

then the analytic characterization of the *limaçon domain* $\mathbb{L}_s(\mathbb{D})$ is given by the inclusion relation (see [13] inclusions (9) and (10))

$$\left\{ w \in \mathbb{C} : |w - 1| < 1 - (1 - |s|)^2 \right\} \subset \mathbb{L}_s(\mathbb{D})$$
$$\subset \left\{ w \in \mathbb{C} : |w - 1| < (1 + |s|)^2 - 1 \right\}.$$

In 1991 Chakrabarti and Jagannathan [5] introduced the concept of (p, q)–calculus in order to generalize or unify several forms of q–oscillator algebras. In the last three decades, applications of the q–calculus have been studied and investigated extensively. Inspired and motivated by these applications many researchers (for example [1], [4]) have developed the theory of quantum calculus based on two-parameter (p,q)–integer which is used efficiently in many fields such as difference equations, Lie group, hypergeometric series, physical sciences, etc.

The (p,q)–bracket or twin basic number $[n]_{p,q}$ is defined by

$$[n]_{p,q} := \begin{cases} \dfrac{p^n - q^n}{p-q}, & \text{if} \quad q \neq p, \\ np^{n-1}, & \text{if} \quad q = p, \end{cases}$$

where $0 < q \leq p < 1$.

For $0 < q < 1$, the q-bracket $[n]_q$ for $n = 0, 1, 2, \ldots$ is given by $[n]_q := [n]_{1,q}$. The (p,q)*–derivative* of a function f is defined by

$$\mathrm{D}_{p,q} f(z) := \begin{cases} \dfrac{f(pz) - f(qz)}{(p-q)z}, & \text{if} \quad q \neq p, \; z \neq 0, \\ 1, & \text{if} \quad p \neq q, \; z = 0, \\ f'(z), & \text{if} \quad p = q. \end{cases}$$

In particular, $\mathrm{D}_{p,q} z^n = [n]_{p,q} z^{n-1}$, therefore, for a function $f \in \mathcal{A}$ of the form (1.1) the (p,q)–derivative operator is given by

$$\mathrm{D}_{p,q} f(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1}, \; z \in \mathbb{D}.$$

In the univalent function theory many extensive studies were given to estimate the upper bounds of the *Hankel determinants*, and for further reading one may refer to [15], [16], [18]. The closer connection with the Hankel determinants are the *Toeplitz determinants*. A Toeplitz determinant can be thought of as an "upside-down" Hankel determinant, in that Hankel determinant have constant entries along the reverse diagonal, whereas Toeplitz matrices have constant entries along the diagonal. In recent past, many researchers have focussed on finding sharp estimates for second and third order Toeplitz determinants [10], [7], etc.

Thomas and Halim [19] defined the *symmetric Toeplitz determinant* $\mathrm{T}_m(n)$ by

$$\mathrm{T}_m(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_n & \cdots & a_{n+m-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+m-1} & a_{n+m-2} & \cdots & a_n \end{vmatrix},$$

and in particular

$$\mathrm{T}_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \quad \mathrm{T}_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}.$$

For a good summary of the applications of Toeplitz matrices to the wide range of areas of pure and applied mathematics, one can refer to [20].

The *logarithmic coefficients* $\gamma_n := \gamma_n(f)$, $n \geq 1$, for a function $f \in \mathcal{S}$ of the form (1.1) play an important role in Milin's conjecture [14] and Brennan's conjecture [12], and can also be used to find estimations for the coefficients of an inverse function. It is given by the power series representation (see [14, p. 53])

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n, \ z \in \mathbb{D}, \tag{1.4}$$

where the function "log" is considered to the main branch, i.e. $\log 1 = 0$. Differentiating the definition relation (1.4) and then equating the coefficients of z^n, the logarithmic coefficients γ_1 and γ_2 will be given by

$$\gamma_1 = \frac{a_2}{2}, \tag{1.5}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right). \tag{1.6}$$

In the theory of univalent functions the problem of finding the sharp estimates for the logarithmic coefficients for various significant classes have gained a high importance (see, for details, [2], [3]). Recently, S. Giri and S. Kumar [8] initiated the study of Toeplitz determinants whose elements are logarithmic coefficients of $f \in \mathcal{S}$ which is given by

$$\mathcal{T}_{m,n}(\gamma_f) := \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+m-1} \\ \gamma_{n+1} & \gamma_n & \cdots & \gamma_{n+m-2} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+m-1} & \gamma_{n+m-2} & \cdots & \gamma_n \end{vmatrix},$$

thus

$$\mathcal{T}_{2,1}(\gamma_f) = \begin{vmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_1 \end{vmatrix}.$$

In this paper we obtained the estimates of Toeplitz determinants and Toeplitz determinanats of logarithmic coefficients for the subclasses $\mathrm{L}_s\mathcal{S}_p^q$, $\mathrm{L}_s\mathcal{C}_p^q$, and $\mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{S}$, $\mathrm{L}_s\mathcal{C}_p^q \cap \mathcal{S}$, $0 < q \leq p \leq 1$, respectively, defined by post quantum operators which map the open unit disc $\mathbb{D}$ in a domain included in the limaçon domain.

2. The subclasses $\mathrm{L}_s\mathcal{S}_p^q$, $\mathrm{L}_s\mathcal{C}_p^q$ and preliminary results

The new subclasses of $\mathcal{A}$ we will define and investigate extend and are connected with the below subclass functions:

Definition 2.1. [17] Denote by S_S^* the subclass of $\mathcal{A}$ consisting of functions given by (1.1) and satisfying

$$\operatorname{Re} \frac{zf'(z)}{f(z) - f(-z)} > 0, \ z \in \mathbb{D}.$$

These functions introduced by Sakaguchi are called *functions starlike with respect to symmetric points*, and for a function $f \in \mathcal{A}$ the above inequality is a necessary and sufficient condition for f to b e univalent and starlike with respect to symmetrical points in $\mathbb{D}$ (see [17, Theorem 1]).

Like we can see in [13, Lemma 2], the function $\mathbb{L}_s$ defined by (1.3) is starlike with respect to the point $z_0 = 1$ for all $s \in [-1, 1] \setminus \{0\}$, hence is univalent in $\mathbb{D}$. Moreover, if $0 < s \leq 1/\sqrt{2}$ then $\mathbb{L}_s$ has real positive part in $\mathbb{D}$, i.e. $\mathbb{L}_s$ is a *Carathéodory function* (see [13, p. 10]).

Now we define the classes $\mathrm{L}_s\mathcal{S}_p^q$ and $\mathrm{L}_s\mathcal{C}_p^q$ which maps the open unit disc onto the region included in the limaçon domain $\mathbb{L}_s(\mathbb{D})$ as follows:

Definition 2.2. Let $\mathrm{L}_s\mathcal{S}_p^q$ be the subclass of function $f \in \mathcal{A}$ of the form (1.1) and satisfying the condition

$$\frac{2z\mathrm{D}_{p,q}f(z)}{f(z) - f(-z)} \prec \mathbb{L}_s(z),\ 0 < s \leq \frac{1}{\sqrt{2}}.$$

Definition 2.3. Let $\mathrm{L}_s\mathcal{C}_p^q$ be the subclass of $\mathcal{A}$ consisting of the function f of the form (1.1) such that

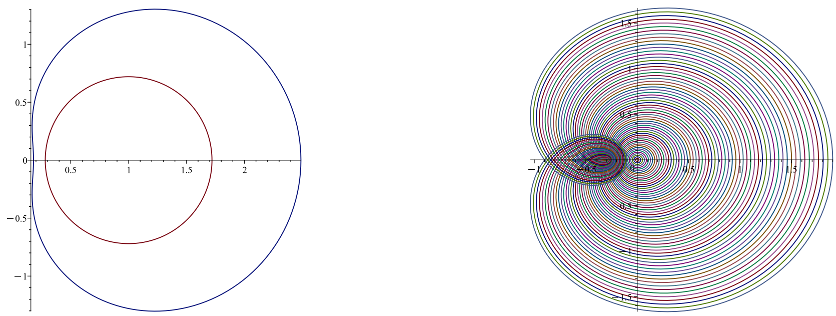
$$\frac{\big(2z\mathrm{D}_{p,q}f(z)\big)'}{\big(f(z) - f(-z)\big)'} \prec \mathbb{L}_s(z),\ 0 < s \leq \frac{1}{\sqrt{2}}.$$

Remark 2.4. The above mentioned classes are not empty, as we will show in the below examples.

(i) Taking $f_*(z) = z + az^2$, $a \in \mathbb{C}$, then

$$\Phi_*(z) := \frac{2z\mathrm{D}_{p,q}f_*(z)}{f_*(z) - f_*(-z)} = 1 + (p+q)az,\ z \in \mathbb{D}.$$

For the values $q = 0.3$, $p = 0.5$, $a = 0.9$, and $s = 1/\sqrt{3}$, like we see in the below Figure 1(A) made with MAPLE™ computer software we have $\Phi_*(\mathbb{D}) \subset \mathbb{L}_{1/\sqrt{3}}(\mathbb{D})$, and because $\Phi_*(0) = \mathbb{L}_{1/\sqrt{3}}(0)$ from the univalence of $\mathbb{L}_{1/\sqrt{3}}$ it follows that $\Phi_*(z) \prec \mathbb{L}_{1/\sqrt{3}}(z)$, i.e. $f_* \in \mathrm{L}_s\mathcal{S}_p^q$ for the previous parameters. Also, the Figure 1(B) shows that the function f_* is not univalent in $\mathbb{D}$ because $f_*(\mathbb{D})$ twice overlaps a subset of $\mathbb{C}$.



(A) The images of $\Phi_*(\partial\mathbb{D})$ (red color) and $\mathbb{L}_{1/\sqrt{3}}(\partial\mathbb{D})$ (blue color)

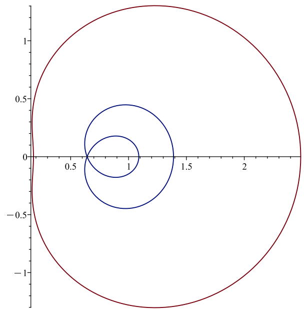
(B) The domain $f_*(\mathbb{D})$

FIGURE 1. Figures for the Remark 2.4(i)

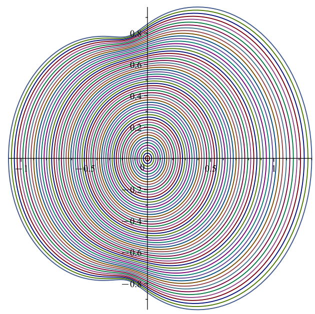
(ii) For $\widehat{f}(z)=z+az^2+bz^3$, $a,b\in\mathbb{C}$, we get

$$\widehat{\Phi}(z):=\frac{2z\mathrm{D}_{p,q}\widehat{f}(z)}{\widehat{f}(z)-\widehat{f}(-z)}=\frac{1+(p+q)az+\left(p^2+pq+q^2\right)bz^2}{1+bz^2},\ z\in\mathbb{D}.$$

If $q=0.85$, $p=0.95$, $a=0.1$, $b=0.2$, and $s=1/\sqrt{3}$, we see in the Figure 2(A) made with MAPLE™ that $\widehat{\Phi}(\mathbb{D})\subset\mathbb{L}_{1/\sqrt{3}}(\mathbb{D})$, and from $\widehat{\Phi}(0)=\mathbb{L}_{1/\sqrt{3}}(0)$ and the univalence of $\mathbb{L}_{1/\sqrt{3}}$ we have $\widehat{\Phi}(z)\prec\mathbb{L}_{1/\sqrt{3}}(z)$, that is $\widehat{\Phi}\in\mathrm{L}_s\mathcal{S}_p^q$ for this choice of the parameters. Moreover, from this figure we wee that $\widehat{\Phi}$ is not univalent in $\mathbb{D}$, while the Figure 2(B) shows that $\widehat{f}$ is univalent in $\mathbb{D}$.



(A) The images of $\widehat{\Phi}(\partial\mathbb{D})$ (blue color) and $\mathbb{L}_{1/\sqrt{3}}(\partial\mathbb{D})$ (red color)



(B) The domain $\widehat{f}(\mathbb{D})$

FIGURE 2. Figures for the Remark 2.4(ii)

(iii) Using the above notations, and

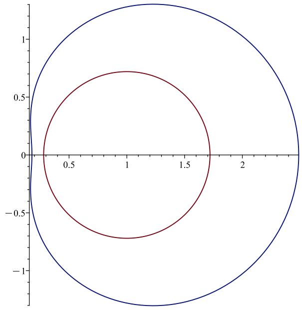
$$\Psi_*(z):=\frac{\left(2z\mathrm{D}_{p,q}f(z)\right)'}{\left(f(z)-f(-z)\right)'}=1+2(p+q)az,\ z\in\mathbb{D}.$$

for $q=0.15$, $p=0.25$, $a=0.9$, and $s=1/\sqrt{3}$, the Figure 3(A) made with MAPLE™ computer software shows that $\Psi_*(\mathbb{D})\subset\mathbb{L}_{1/\sqrt{3}}(\mathbb{D})$, and because $\Psi_*(0)=\mathbb{L}_{1/\sqrt{3}}(0)$ from the univalence of $\mathbb{L}_{1/\sqrt{3}}$ it follows $\Psi_*(z)\prec\mathbb{L}_{1/\sqrt{3}}(z)$, i.e. $f_*\in\mathrm{L}_s\mathcal{C}_p^q$ for these values of the parameters. The Figure 3(B) shows that the function f_* is not univalent in $\mathbb{D}$ since there exists a subset of $\mathbb{C}$ that's twice overlapped by $f_*(\mathbb{D})$.

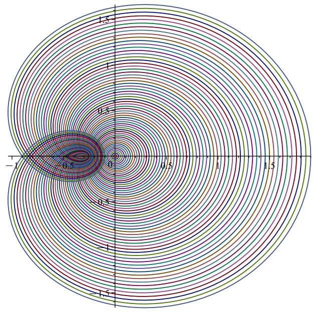
(iv) Considering the function $\widehat{f}(z)=z+az^2+bz^3$, $a,b\in\mathbb{C}$, we get

$$\widehat{\Psi}(z):=\frac{\left(2z\mathrm{D}_{p,q}f(z)\right)'}{\left(f(z)-f(-z)\right)'}=\frac{1+2(p+q)az+3\left(p^2+pq+q^2\right)bz^2}{1+3bz^2},\ z\in\mathbb{D}.$$

For $q=0.4$, $p=0.5$, $a=0.25$, $b=0.2$, and $s=1/\sqrt{3}$, we see in the Figure 4(A) made with MAPLE™ that $\widehat{\Psi}(\mathbb{D})\subset\mathbb{L}_{1/\sqrt{3}}(\mathbb{D})$. Using that $\widehat{\Psi}(0)=\mathbb{L}_{1/\sqrt{3}}(0)$ together with the univalence of $\mathbb{L}_{1/\sqrt{3}}$ we have $\widehat{\Psi}(z)\prec\mathbb{L}_{1/\sqrt{3}}(z)$, that is $\widehat{\Psi}\in\mathrm{L}_s\mathcal{S}_p^q$ for these



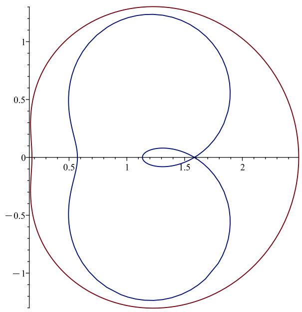
(A) The images of $\Phi_*(\partial\mathbb{D})$ (red color) and $\mathbb{L}_{1/\sqrt{3}}(\partial\mathbb{D})$ (blue color)



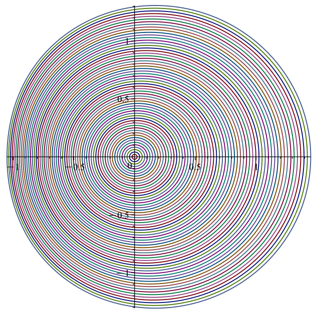
(B) The domain $f_*(\mathbb{D})$

FIGURE 3. Figures for the Remark 2.4(iii)

choice of the parameters. Moreover, from this figure we wee that $\widehat{\Psi}$ is not univalent in $\mathbb{D}$, and from the Figure 4(B) we see that $\widehat{f}$ is univalent in $\mathbb{D}$.



(A) The images of $\widehat{\Psi}(\partial\mathbb{D})$ (blue color) and $\mathbb{L}_{1/\sqrt{3}}(\partial\mathbb{D})$ (red color)



(B) The domain $\widehat{f}(\mathbb{D})$

FIGURE 4. Figures for the Remark 2.4(iv)

(v) Concluding, the examples given in the Remark 2.4(i)–(iv) show that $\mathrm{L}_s\mathcal{S}_p^q \neq \emptyset$ and $\mathrm{L}_s\mathcal{C}_p^q \neq \emptyset$. From the examples of the Remark 2.4(i) and (iii) it follows that $\mathrm{L}_s\mathcal{S}_p^q \not\subset \mathcal{S}$ and $\mathrm{L}_s\mathcal{C}_p^q \not\subset \mathcal{S}$. In addition, the examples of the Remark 2.4(ii) and (iv) show that the corresponding functions of the form f_* and $\widehat{f}$ belong to $\mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{S}$ and $\mathrm{L}_s\mathcal{C}_p^q \cap \mathcal{S}$, respectively, i.e. $\mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{S} \neq \emptyset$ and $\mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{C} \neq \emptyset$. These above comments are very important for the motivations of the results presented in the Sections 3 and 4.

In our investigations we will use the next lemmas:

Lemma 2.5. [11, Lemma 2.1] *If the function $w \in \mathcal{B}$ is of the form* (1.2)*, then for some complex numbers ξ and ζ such that $|\xi| \leq 1$ and $|\zeta| \leq 1$, we have*

$$\begin{aligned} &w_2 = \xi\left(1 - w_1^2\right), \text{ and} \\ &w_3 = \left(1 - w_1^2\right)\left(1 - |\xi|^2\right)\zeta - w_1\left(1 - w_1^2\right)\xi^2. \end{aligned}$$

Lemma 2.6. [9, p. 3, Lemma 1], [6] *If the function $w \in \mathcal{B}$ is of the form* (1.2)*, then the sharp estimate $|w_n| \leq 1$ holds for $n \geq 1$.*

3. Symmetric Toeplitz determinants of the coefficients for the classes $\mathrm{L}_s\mathcal{S}_p^q$ and $\mathrm{L}_s\mathcal{C}_p^q$

Now we will give upper bounds for some symmetric Toeplitz determinants for the functions belonging to the above defined classes $\mathrm{L}_s\mathcal{S}_p^q$ and $\mathrm{L}_s\mathcal{C}_p^q$, emphasizing that for $|\mathrm{T}_2(2)|$ the results are sharp.

Theorem 3.1. *If the function $f \in \mathrm{L}_s\mathcal{S}_p^q$ has the form* (1.1)*, then*

$$|\mathrm{T}_2(2)| \leq \frac{s^2(s+4)^2}{\left([3]_{p,q} - 1\right)^2} + \frac{4s^2}{\left([2]_{p,q}\right)^2},$$

and this inequality is sharp (i.e. the best possible).

Proof. Assuming that $f \in \mathrm{L}_s\mathcal{S}_p^q$, according to the definition of the subordination there exists a function $w \in \mathcal{B}$ of the form (1.2) such that

$$\frac{2z\mathrm{D}_{p,q}f(z)}{f(z) - f(-z)} = \left(1 + sw(z)\right)^2, \ z \in \mathbb{D}. \tag{3.1}$$

Since (3.1) is equivalent to

$$2z\mathrm{D}_{p,q}f(z) = \left(f(z) - f(-z)\right)\left(1 + sw(z)\right)^2, \ z \in \mathbb{D},$$

expanding in Taylor series the both sides of the above relation and equating the corresponding terms we have

$$\begin{aligned} &z + z^2[2]_{p,q}a_2 + z^3a_3[3]_{p,q} + z^4a_4[4]_{p,q} + \cdots = \\ &z + 2sw_1z^2 + z^3\left(a_3 + 2sw_2 + s^2w_1^2\right) + 2z^4\left(sw_1a_3 + sw_3 + w_1w_2\right) + \ldots, \end{aligned}$$

thus

$$a_2 = \frac{2sw_1}{[2]_{p,q}} = \frac{2sw_1}{t_2}, \tag{3.2}$$

$$a_3 = \frac{2sw_2 + s^2w_1^2}{[3]_{p,q} - 1} = \frac{2sw_2 + s^2w_1^2}{t_3 - 1}, \tag{3.3}$$

where, for simplicity, we use the notation $t_n{:=}[n]_{p,q}$.

It follows that

$$|\mathrm{T}_2(2)| = \left|a_3^2 - a_2^2\right| = \left|\left(\frac{2sw_2 + s^2w_1^2}{t_3 - 1}\right)^2 - \left(\frac{2sw_1}{t_2}\right)^2\right|, \tag{3.4}$$

and rewriting w_2 in terms of w_1 from Lemma 2.5, we get

$$|\mathrm{T}_2(2)| = \left|\left(\frac{2s\left(1-w_1^2\right)\xi + s^2w_1^2}{t_3-1}\right)^2 - \left(\frac{2sw_1}{t_2}\right)^2\right|. \tag{3.5}$$

From the relation (3.5), using the triangle's inequality and the fact that $s>0$ we get first that

$$\begin{aligned}|\mathrm{T}_2(2)| &= \left|\frac{4s^2\left(1-w_1^2\right)^2\xi^2 + s^4w_1^4 + 4s^3\left(1-w_1^2\right)\xi\, w_1^2}{\left(t_3-1\right)^2} - \frac{4s^2w_1^2}{t_2^2}\right| \\ &\leq \frac{4s^2\left|1-w_1^2\right|^2|\xi|^2 + s^4|w_1|^4 + 4s^3\left|1-w_1^2\right|\,|\xi|\,|w_1|^2}{\left(t_3-1\right)^2} + \frac{4s^2|w_1|^2}{t_2^2}. \end{aligned} \tag{3.6}$$

Denoting $x := |w_1|$ and $y := |\xi|$, then $x,\, y \in [0,1]$, and

$$\left|1-w_1^2\right| \leq 1+x^2, \quad \left|1-w_1^2\right|^2 \leq \left(1+x^2\right)^2, \tag{3.7}$$

if we combine the inequalities (3.7) with (3.6) it follows

$$|\mathrm{T}_2(2)| \leq \frac{4s^2\left(1+x^2\right)^2 y^2 + s^4x^4 + 4s^3\left(1+x^2\right)\, y\, x^2}{\left(t_3-1\right)^2} + \frac{4s^2x^2}{t_2^2} =: h(x,y). \tag{3.8}$$

Since

$$\frac{\partial}{\partial y}h(x,y) = \frac{8s^2\left(x^2+1\right)^2 y + 4s^3\left(x^2+1\right)x^2}{\left(t_3-1\right)^2} \geq 0,\ (x,y)\in[0,1]\times[0,1],$$

we obtain that for any $x\in[0,1]$ we have

$$\max\left\{h(x,y) : y\in[0,1]\right\} = h(x,1) =: g(x)$$

and consequently, from (3.8) we get

$$|\mathrm{T}_2(2)| \leq \frac{4s^2\left(1+x^2\right)^2 + s^4x^4 + 4s^3\left(1+x^2\right)\, x^2}{\left(t_3-1\right)^2} + \frac{4s^2x^2}{t_2^2} = g(x). \tag{3.9}$$

Using the fact that

$$g'(x) = \frac{8x\left[\dfrac{(s+2)\left(s\,x^2+2x^2+2\right)t_2^2}{2} + \left(t_3-1\right)^2\right]s^2}{\left(t_3-1\right)^2 t_2^2} \geq 0,\ x\in[0,1],$$

we have that g is an increasing function on $[0,1]$. Therefore, the inequality (3.9) leads us to

$$|\mathrm{T}_2(2)| \leq g(1) = \frac{s^2(s+4)^2}{\left(t_3-1\right)^2} + \frac{4s^2}{t_2^2},\ x\in[0,1],$$

that proves the required inequality.

To prove the sharpness of our result, let consider the function $f \in \mathcal{A}$ given by (3.1) with $w(z) = iz - 2z^2$. Since $w_1 = i$, $w_2 = -2$, using the relation (3.4) we have

$$|\mathrm{T}_2(2)| = \left| \left(\frac{-4s - s^2}{t_3 - 1} \right)^2 + \left(\frac{2s}{t_2} \right)^2 \right| = \frac{s^2(s+4)^2}{(t_3 - 1)^2} + \frac{4s^2}{t_2^2},$$

which shows the sharpness of our inequality. □

Theorem 3.2. *If the function* $f \in \mathrm{L}_s\mathcal{S}_p^q$ *has the form* (1.1), *then*

$$|\mathrm{T}_3(1)| \leq 1 + \frac{8s^2}{\left([2]_{p,q}\right)^2} + \frac{8s^3(s+4)}{\left([2]_{p,q}\right)^2 \left|[3]_{p,q} - 1\right|} + \frac{s^2(s+4)^2}{\left([3]_{p,q} - 1\right)^2}.$$

Proof. Using the same techniques and notations like in the proof of Theorem 3.1 we have

$$\begin{aligned} |\mathrm{T}_3(1)| &= \left|1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2\right| \\ &= \left| 1 - 2\left(\frac{2sw_1}{t_2}\right)^2 + 2\left(\frac{2sw_1}{t_2}\right)^2 \cdot \frac{2sw_2 + s^2 w_1^2}{t_3 - 1} - \left(\frac{2sw_2 + s^2w_1^2}{t_3 - 1}\right)^2 \right|. \end{aligned}$$

From Lemma 2.5, rewriting the expression w_2 in terms of w_1 the above relation leads to

$$\begin{aligned} |\mathrm{T}_3(1)| = \Bigg| 1 &- 2\left(\frac{2sw_1}{t_2}\right)^2 + 2\left(\frac{2sw_1}{t_2}\right)^2 \cdot \frac{2s\left(1 - w_1^2\right)\xi + s^2 w_1^2}{t_3 - 1} \\ &- \frac{4s^2\left(1 - w_1^2\right)^2 \xi^2 + s^4 w_1^4 + 4s^3 w_1^2 \left(1 - w_1^2\right)\xi}{(t_3 - 1)^2} \Bigg|. \end{aligned} \tag{3.10}$$

Letting $x := |w_1|$ and $y := |\xi|$, then $x,\, y \in [0,1]$, and applying the triangle's inequality in the right hand side of (3.10), since $s > 0$ we obtain

$$\begin{aligned} |\mathrm{T}_3(1)| \leq 1 &+ \frac{8s^2x^2}{t_2^2} + \frac{8s^2x^2\left[2s\left(1 + x^2\right)y + s^2x^2\right]}{t_2^2\, |t_3 - 1|} \\ &+ \frac{4s^2\left(1 + x^2\right)^2 y^2 + s^4x^4 + 4s^3x^2\left(1 + x^2\right)y}{(t_3 - 1)^2} =: q(x,y). \end{aligned} \tag{3.11}$$

A simple computation shows that for all $(x,y) \in [0,1] \times [0,1]$ we have

$$\frac{\partial}{\partial y} q(x,y) = \frac{16s^3x^2\left(x^2 + 1\right)}{t_2^2 |t_3 - 1|} + \frac{8s^2\left(x^2 + 1\right)^2 y + 4s^3x^2\left(x^2 + 1\right)}{(t_3 - 1)^3} \geq 0,$$

therefore, for any $x \in [0,1]$ we have

$$\begin{aligned} \max\left\{q(x,y) : y \in [0,1]\right\} = q(x,1) = 1 &+ \frac{8s^2x^2}{t_2^2} + \frac{8s^2x^2\left[2s\left(1 + x^2\right) + s^2x^2\right]}{t_2^2\, |t_3 - 1|} \\ &+ \frac{4s^2\left(1 + x^2\right)^2 + s^4x^4 + 4s^3x^2\left(1 + x^2\right)}{(t_3 - 1)^2} =: t(x), \end{aligned}$$

hence, from (3.11) it follows

$$|\mathrm{T}_3(1)| \le t(x),\ x \in [0,1]. \tag{3.12}$$

Moreover, since

$$t'(x) = \frac{16s^2x}{t_2^2} + \frac{16s^2x\left[2s\left(x^2+1\right)+s^2x^2\right]}{t_2^2|t_3-1|} + \frac{8s^2x^2\left(2s^2x+4sx\right)}{t_2^2\,|t_3-1|}$$
$$+\frac{16s^2\left(x^2+1\right)x+4s^4x^3+8s^3x\left(x^2+1\right)+8s^3x^3}{\left(t_3-1\right)^3} \ge 0,\ x\in[0,1],$$

the function t is increasing on $[0,1]$, and from (3.12) we deduce that

$$|\mathrm{T}_3(1)| \le t(1) = 1 + \frac{8s^2}{t_2^2} + \frac{8s^3(s+4)}{t_2^2\,|t_3-1|} + \frac{s^2(s+4)^2}{\left(t_3-1\right)^2},$$

which represents the required inequality. □

Theorem 3.3. *If the function $f \in \mathrm{L}_s\mathcal{C}_p^q$ has the form* (1.1), *then*

$$|\mathrm{T}_2(2)| \le \frac{s^2(s+4)^2}{9\big([3]_{p,q}-1\big)^2} + \frac{s^2}{\big([2]_{p,q}\big)^2},$$

and this inequality is sharp (i.e. the best possible).

Proof. For the function $f \in \mathrm{L}_s\mathcal{C}_p^q$, using the definition of the subordination there exists a function $w(z) = w_1z + w_2z^2 + \cdots \in \mathcal{B}$, $z \in \mathbb{D}$, such that

$$\frac{\big(2z\mathrm{D}_{p,q}f(z)\big)'}{\big(f(z)-f(-z)\big)'} = \big(1+sw(z)\big)^2,\ z\in\mathbb{D}. \tag{3.13}$$

The relation (3.13) could be written in the form

$$\big(2z\mathrm{D}_{p,q}f(z)\big)' = \big(f(z)-f(-z)\big)'\big(1+sw(z)\big)^2,\ z\in\mathbb{D},$$

and expanding in Taylor series both sides of this equality we get

$$1+2z[2]_{p,q}a_2+3z^2a_3[3]_{p,q}+4z^3a_4[4]_{p,q}+\cdots =$$
$$1+2sw_1z+z^2\left(s^2w_1^2+2sw_2+3a_3\right)+z^3\left(2s^2w_1w_2+2sw_3+6sw_1a_3\right)+\ldots.$$

Equating the corresponding coefficients it follows that

$$a_2 = \frac{sw_1}{[2]_{p,q}}, \tag{3.14}$$

$$a_3 = \frac{2sw_2+s^2w_1^2}{3\big([3]_{p,q}-1\big)}. \tag{3.15}$$

Using Lemma 2.5 it's easy to check that

$$|\mathrm{T}_2(2)| = \left|a_3^2-a_2^2\right| = \left|\frac{4s^2w_2^2+s^4w_1^4+4s^3w_1^2w_2}{9\left(t_3-1\right)^2} - \frac{s^2w_1^2}{t_2^2}\right|$$
$$= \left|\frac{4s^2\left(1-w_1^2\right)^2\xi^2+s^4w_1^4+4s^3w_1^2\left(1-w_1^2\right)\xi}{9\left(t_3-1\right)^2} - \frac{s^2w_1^2}{t_2^2}\right|, \tag{3.16}$$

where we use the previous notation $t_n:=[n]_{p,q}$.

Denoting $x := |w_1|$ and $y := |\xi|$, then $x, y \in [0,1]$, and using the triangle's inequality in the right hand side of the above relation, since $s>0$ we have

$$|\mathrm{T}_2(2)| \leq \frac{4s^2\left(1+x^2\right)^2 y^2 + s^4x^4 + 4s^3\left(1+x^2\right)x^2y}{9\left(t_3-1\right)^2} + \frac{s^2x^2}{t_2^2} =: h(x,y). \tag{3.17}$$

It is easy to see that

$$\frac{\partial}{\partial y}h(x,y) = \frac{4s^2\left(x^2+1\right)\left[(s+2y)\,x^2+2y\right]}{9\left(t_3-1\right)^2} \geq 0,\ (x,y)\in[0,1]\times[0,1],$$

consequently, for each $x\in[0,1]$ we have

$$\max\left\{h(x,y): y\in[0,1]\right\} = h(x,1)$$
$$= \frac{4s^2\left(1+x^2\right)^2 + s^4x^4 + 4s^3\left(1+x^2\right)x^2}{9\left(t_3-1\right)^2} + \frac{s^2x^2}{t_2^2} =: g(x).$$

Combining this last relation with the inequality (3.17) we obtain

$$|\mathrm{T}_2(2)| \leq g(x). \tag{3.18}$$

Since for all $x\in[0,1]$ we have

$$g'(x) = \frac{16s^2\left(x^2+1\right)x + 4s^4x^3 + 8s^3x^3 + 8s^3\left(x^2+1\right)x}{9\left(t_3-1\right)^2} + \frac{2s^2x}{t_2^2} \geq 0,$$

the function g is increasing on $[0,1]$, therefore the inequality (3.18) leads to

$$|\mathrm{T}_2(2)| \leq g(1) = \frac{s^2(s+4)^2}{9\left(t_3-1\right)^2} + \frac{s^2}{t_2^2},$$

and our conclusion is proved.

The inequality is sharp for the function $f\in\mathcal{A}$ given by (3.1) with $w(z) = iz-2z^2$. In this case $w_1 = i$, $w_2 = -2$, and from the relation (3.16) we get

$$|\mathrm{T}_2(2)| = \frac{s^2(s+4)^2}{\left(t_3-1\right)^2} + \frac{4s^2}{t_2^2},$$

which proves the sharpness of our inequality $\square$

Using the same techniques as in the previous theorem, we obtain the next upper bound for $|\mathrm{T}_3(1)|$ if $f\in\mathrm{L}_s\mathcal{C}_p^q$.

Theorem 3.4. *If the function $f\in\mathrm{L}_s\mathcal{C}_p^q$ has the form (1.1), then*

$$|\mathrm{T}_3(1)| \leq 1 + \frac{2s^2}{\left([2]_{p,q}\right)^2} + \frac{2s^3(s+4)}{3\left([2]_{p,q}\right)^2\left|[3]_{p,q}-1\right|} + \frac{s^2(s+4)^2}{9\left([3]_{p,q}-1\right)^2}.$$

Proof. With the same techniques and notations as in the proof of the previous theorem we have

$$|\mathrm{T}_3(1)| = \left|1 - 2\,\frac{s^2w_1^2}{t_2^2} + 2\,\frac{s^2w_1^2}{t_2^2}\cdot\frac{s^2w_1^2+2sw_2}{3\left(t_3-1\right)} - \frac{s^4w_1^4+4s^2w_2^2+4s^3w_1^2w_2}{9\left(t_3-1\right)^2}\right|.$$

Rewriting the expression w_2 in terms of w_1 like in Lemma 2.5, applying the triangle's inequality, denoting $x = |w_1| \leq 1$, $y = |\xi| \leq 1$, and using that $s > 0$ we get

$$|\mathrm{T}_3(1)| \leq 1 + \frac{2s^2x^2}{t_2^2} + \frac{2s^2x^2\left[s^2x^2 + 2s\left(1+x^2\right)y\right]}{3t_2^2\left|t_3 - 1\right|}$$
$$+\frac{s^4x^4 + 4s^2\left(1+x^2\right)^2y^2 + 4s^3x^2\left(1+x^2\right)y}{9\left(t_3-1\right)^2} =: p(x,y). \tag{3.19}$$

It follows that

$$\frac{\partial}{\partial y}p(x,y) = \frac{4s^3x^2\left(x^2+1\right)}{3t_2^2|t_3-1|} + \frac{4s\left[2sy\left(x^2+1\right)+s^2x^2\right]\left(x^2+1\right)}{9\left(t_3-1\right)^2} \geq 0,$$
$$(x,y) \in [0,1]\times[0,1],$$

hence, for each $x \in [0,1]$ we have

$$\max\left\{p(x,y) : y \in [0,1]\right\} = p(x,1) = 1 + \frac{2s^2x^2}{t_2^2} + \frac{2s^2x^2\left[s^2x^2 + 2s\left(1+x^2\right)\right]}{3t_2^2\left|t_3-1\right|}$$
$$+\frac{s^4x^4 + 4s^2\left(1+x^2\right)^2 + 4s^3x^2\left(1+x^2\right)}{9\left(t_3-1\right)^2} =: q(x). \tag{3.20}$$

Using that

$$q'(x) = \frac{8s^3x\left(x^2+1\right)}{3t_2^2|t_3-1|} + \frac{8s^3x^3}{3t_2^2|t_3-1|} + \frac{\left(2s^2x + 4syx\right)s\left(x^2+1\right)}{9\left(t_3-1\right)^2}$$
$$+\frac{8\left[2sy\left(x^2+1\right)+s^2x^2\right]sx}{9\left(t_3-1\right)^2} \geq 0,\ x \in [0,1],$$

the function q is increasing on $[0,1]$, and from the inequalities (3.19) and (3.20) we conclude that

$$|\mathrm{T}_3(1)| \leq q(1) = 1 + \frac{2s^2}{t_2^2} + \frac{2s^3(s+4)}{3t_2^2\left|t_3-1\right|} + \frac{s^2(s+4)^2}{9\left(t_3-1\right)^2}.$$

$\square$

4. Symmetric Toeplitz determinants of the logarithmic coefficients for the classes $\mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{S}$ and $\mathrm{L}_s\mathcal{C}_p^q \cap \mathcal{S}$

In this section we find the estimates of initial two logarithmic coefficients and then the estimate of symmetric Toeplitz determinants $\mathcal{T}_{2,1}\left(\gamma_f\right)$ of logarithmic coefficients for the subclasses $\mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{S}$ and $\mathrm{L}_s\mathcal{C}_p^q \cap \mathcal{S}$.

Theorem 4.1. *If the function $f \in \mathrm{L}_s\mathcal{S}_p^q \cap \mathcal{S}$ has the form* (1.1) *and the logarithmic coefficients are given by* (1.4)*, then*

$$|\gamma_1| \leq \frac{s}{[2]_{p,q}} \quad \text{and} \quad |\gamma_2| \leq \frac{s(s+4)}{2\left|[3]_{p,q}-1\right|} + \frac{s^2}{\left([2]_{p,q}\right)^2}.$$

Proof. Replacing the values of a_2 and a_3 given by (3.2) and (3.3) in (1.5) and (1.6), using the notation $t_n:=[n]_{p,q}$, from Lemma 2.6 we obtain

$$|\gamma_1| = \left|\frac{sw_1}{t_2}\right| \leq \frac{s}{t_2} = \frac{s}{[2]_{p,q}}.$$

In addition, using Lemma 2.5 we get

$$|\gamma_2| = \frac{1}{2}\left|\frac{2sw_2+s^2w_1^2}{t_3-1} - \frac{2s^2w_1^2}{t_2^2}\right| = \frac{1}{2}\left|\frac{2s\left(1-w_1^2\right)\xi+s^2w_1^2}{t_3-1} - \frac{2s^2w_1^2}{t_2^2}\right|,$$

where $|\xi| \leq 1$. Letting $x := |w_1|$ and $y := |\xi|$, then $x, \xi \in [0,1]$ and using the triangle's inequality in the above relation together with $s > 0$ we obtain

$$|\gamma_2| \leq \frac{2s\left(1+x^2\right)y+s^2x^2}{2\,|t_3-1|} + \frac{s^2x^2}{t_2^2} =: F(x,y). \tag{4.1}$$

It follows that

$$\frac{\partial}{\partial y}F(x,y) = \frac{s\left(1+x^2\right)}{|t_3-1|} > 0,\ (x,y) \in [0,1]\times[0,1],$$

hence, for each $x \in [0,1]$ we have

$$\max\left\{F(x,y) : y \in [0,1]\right\} = F(x,1) = \frac{2s\left(1+x^2\right)+s^2x^2}{2\,|t_3-1|} + \frac{s^2x^2}{t_2^2} =: r(x). \tag{4.2}$$

From the fact

$$r'(x) = \frac{sx(s+2)}{|t_3-1|} + \frac{2s^2x}{t_2^2} \geq 0,\ x \in [0,1],$$

the function r is increasing on $[0,1]$, and from (4.1) and (4.2) we conclude that

$$|\gamma_2| \leq r(1) = \frac{4s+s^2}{2\,|t_3-1|} + \frac{s^2}{t_2^2},$$

which proves our second inequality. □

Theorem 4.2. *If the function $f \in \mathrm{L}_s\mathcal{C}_p^q \cap \mathcal{S}$ has the form* (1.1) *and the logarithmic coefficients are given by* (1.4), *then*

$$|\gamma_1| \leq \frac{s}{2\,[2]_{p,q}} \quad \textit{and} \quad |\gamma_2| \leq \frac{s(s+4)}{6\big|[3]_{p,q}-1\big|} + \frac{s^2}{4\big([2]_{p,q}\big)^2}.$$

Proof. Using the values of a_2 and a_3 given by (3.14) and (3.15), from (1.5) and (1.6), using Lemma 2.6 we obtain

$$|\gamma_1| = \left|\frac{sw_1}{2t_2}\right| \leq \frac{s}{2|t_2|} \quad \text{and} \quad |\gamma_2| = \frac{1}{2}\left|\frac{2sw_2+s^2w_1^2}{3\,(t_3-1)} - \frac{s^2w_1^2}{2t_2^2}\right|.$$

Rewriting the expression of w_2 in terms of w_1 according to Lemma 2.5, using the triangle's inequality in the above last relation, and the notations $x := |w_1|$, $y := |\xi|$, with $x, \xi \in [0,1]$, since $s > 0$ we obtain

$$|\gamma_2| \leq \frac{2s\left(1+x^2\right)y+s^2x^2}{6\,|t_3-1|} + \frac{s^2x^2}{4t_2^2} =: G(x,y). \tag{4.3}$$

Therefore

$$\frac{\partial}{\partial y}G(x,y)=\frac{s\left(1+x^2\right)}{3\left|t_3-1\right|}>0,\ (x,y)\in[0,1]\times[0,1],$$

hence, for each $x\in[0,1]$ we have

$$\max\left\{G(x,y):y\in[0,1]\right\}=G(x,1)=\frac{2s\left(1+x^2\right)+s^2x^2}{6\left|t_3-1\right|}+\frac{s^2x^2}{4t_2^2}=:k(x). \tag{4.4}$$

Since

$$k'(x)=\frac{sx(sx+2)}{3\left|t_3-1\right|}+\frac{s^2x}{2t_2^2}\geq 0,\ x\in[0,1],$$

the function k is increasing on $[0,1]$, and combining (4.3) with (4.4) it follows

$$\left|\gamma_2\right|\leq k(1)=\frac{s(s+4)}{6\left|t_3-1\right|}+\frac{s^2}{4t_2^2},$$

and the proof is complete. □

The following two results, where we determined the upper bounds for the Toeplitz determinant $\left|\mathcal{T}_{2,1}\left(\gamma_f\right)\right|$ for the classes $\mathrm{L}_s\mathcal{S}_p^q\cap\mathcal{S}$ and $\mathrm{L}_s\mathcal{C}_p^q\cap\mathcal{S}$ are immediately consequences of the previous two theorems.

Corollary 4.3. *For the class* $\mathrm{L}_s\mathcal{S}_p^q\cap\mathcal{S}$ *the next inequality holds:*

$$\left|\mathcal{T}_{2,1}\left(\gamma_f\right)\right|\leq\left(\frac{s}{[2]_{p,q}}\right)^2+\left(\frac{s(s+4)}{2\left|[3]_{p,q}-1\right|}+\frac{s^2}{\left([2]_{p,q}\right)^2}\right)^2.$$

Proof. Since

$$\left|\mathcal{T}_{2,1}\left(\gamma_f\right)\right|=\left|\gamma_1^2-\gamma_2^2\right|\leq\left|\gamma_1^2\right|+\left|\gamma_2^2\right|$$

from the inequalities of Theorem 4.1 we get

$$\left|\mathcal{T}_{2,1}\left(\gamma_f\right)\right|\leq\left(\frac{s}{t_2}\right)^2+\left(\frac{s(s+4)}{2\left|t_3-1\right|}+\frac{s^2}{t_2^2}\right)^2.$$

□

Similarly, using the inequalities obtained in Theorem 4.2 it's easy to prove the next result:

Corollary 4.4. *For the class* $\mathrm{L}_s\mathcal{C}_p^q\cap\mathcal{S}$ *the next inequality holds:*

$$\left|\mathcal{T}_{2,1}\left(\gamma_f\right)\right|\leq\left(\frac{s}{2\,[2]_{p,q}}\right)^2+\left(\frac{s(s+4)}{6\left|[3]_{p,q}-1\right|}+\frac{s^2}{4\left([2]_{p,q}\right)^2}\right)^2.$$

5. Concluding remarks

The quantum calculus is one of the important tools in many area of mathematics, physics and in the areas of ordinary fractional calculus, optimal control problems, quantum physics, operator theory, and q–transform analysis, and in this paper we made a connection with some subclasses of analytic functions.

In addition, the logarithmic coefficients play an important role for different estimates in the theory of univalent functions. Many researchers have found the upper bounds for the second and third order Toeplitz determinants and logarithmic coefficients for various subclasses of analytic function. The present investigation deals with the subclasses of symmetric function using the (p,q)–calculus for some functions defined by subordinations to the limaçon domain, and we determined upper bounds for some special symmetric Toeplitz determinants containing the coefficients and the logarithmic coefficients of the functions belonging to these classes. We obtained bounds for the second and third order Toeplitz determinants and Toeplitz determinants for logarithmic coefficients for the classes $\mathrm{L}_s\mathcal{S}_p^q$, $\mathrm{L}_s\mathcal{C}_p^q$, and $\mathrm{L}_s\mathcal{S}_p^q\cap\mathcal{S}$, $\mathrm{L}_s\mathcal{C}_p^q\cap\mathcal{S}$, respectively, defined by the post-quantum operators and subordinated to $\mathbb{L}_s$ function.

We hope that these results could be important in several fields related to mathematics, engineering, science and technology, and we encourage the researchers to find the sharp estimates for third order Toeplitz determinants and Toeplitz determinants for logarithmic coefficients.

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Generalization of Jack's lemma for functions with fixed initial coefficient and its applications

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Abstract. In this paper, by using the theory of differential subordination, we will generalize Jack's lemma for functions with fixed initial coefficient. Then extensions of the well-known open-door lemma for analytic and meromorphic functions with fixed initial coefficient are given. Also we consider some applications of the extension of Jack's lemma.

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1. Introduction and preliminaries

Let $\mathcal{H}$ denote the set of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We define

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\},$$

where n is a positive integer number and $a \in \mathbb{C}$. Suppose $n \in \mathbb{N}$, we introduce the subclass $\mathcal{A}_n$ of $\mathcal{H}$ as follows:

$$\mathcal{A}_n = \{f \in \mathcal{H} : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots\}.$$

In addition to, in particular, we set $\mathcal{A}_1 = \mathcal{A}$. Also we define the subclass $\mathcal{S}$ of $\mathcal{A}$ consisting of univalent functions in the open unit disk $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be starlike of order $0 \leq \gamma < 1$, written $f \in S^*(\gamma)$, if it satisfies

$$\mathfrak{Re}\frac{zf'(z)}{f(z)} > \gamma \qquad (z \in \mathbb{U}).$$

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Especially we set $\mathcal{S}^*(0) \equiv \mathcal{S}^*$. Now for analytic functions in $\mathbb{U}$ with fixed initial coefficient, we define the class $\mathcal{H}_\beta[a,n]$ as follows:

$$\mathcal{H}_\beta[a,n] = \{f \in \mathcal{H} : f(z) = a + \beta z^n + a_{n+1}z^{n+1} + \dots\},$$

where n is a positive integer number, $a \in \mathbb{C}$ and $\beta \in \mathbb{C}$ is a fixed number. Moreover we assume

$$\mathcal{A}_{n,b} = \{f \in \mathcal{H} : f(z) = z + bz^{n+1} + a_{n+2}z^{n+2} + \dots\},$$

where n is a positive integer number and $b \in \mathbb{C}$ is a fixed number. Also we set $\mathcal{A}_b = \mathcal{A}_{1,b}$. Let f and g be in $\mathcal{H}$. We say that the function f is subordinate to g, denoted by $f \prec g$, if there exists an analytic function in $\mathbb{U}$ as ω, with $\omega(0) = 0$ and $|\omega(z)| \le |z| < 1$, such that $f(z) = g(\omega(z))$. Moreover if g is an univalent function in $\mathbb{U}$, then $f \prec g$ if and only if $f(0) = 0$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

It is important to note that coefficients of analytic functions play important role in geometric functions theory. For example, the bound on the second coefficient of an univalent function leads to well-known results such as growth, distortion and covering theorems (see [8]). Recently the subject of second order differential subordination for analytic functions with fixed initial coefficient was considered by Ali et al.[2]. Then in the papers [7, 6, 9] the authors by applying first order differential subordination for functions with fixed initial coefficient related to univalent functions, obtained some good results.

Furthermore in [1], the problem of radius of starlikeness for analytic functions with fixed second coefficient is discussed. Also, Amani et al., [3, 4] have obtained some results for functions with fixed initial coefficient.

Motivated by [3] and [4], in this paper we extend the famous Jake's Lemma for analytic functions with fixed second coefficient.

We organize the contents as follows. In Section 2, we will bring extension of Jack's Lemma and open-door lemma for analytic and meromophic functions with fixed initial coefficient and then we include some corollaries from them. In Section 3, we apply the results in the sections 2, for obtaining some sufficient conditions for starlikeness and carathedory functions.

In the continuation of work, for proving main results, we require to express a definition and a basic lemma.

Definition 1.1. (see [8]) Let Q denote the set of functions q that are analytic and injective on $\overline{\mathbb{U}}\backslash E(q)$, where

$$E(q) := \left\{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} q(z) = \infty\right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(q)$.

Lemma 1.2. *(see [2]) Let $q \in Q$ with $q(0) = a$ and $p \in \mathcal{H}_c[a,n]$ with $p(z) \not\equiv a$. If there exist a point $z_0 \in \mathbb{U}$ such that $p(z_0) \in q(\partial\mathbb{U})$ and $p(\{z : |z| < |z_0|\}) \subset q(\mathbb{U})$ then*

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0) \tag{1.1}$$

and

$$\mathfrak{Re}\left\{1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right\} \geq m\mathfrak{Re}\left\{1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right\} \tag{1.2}$$

where $q^{-1}(p(z_0)) = \zeta_0 = e^{i\theta_0}$ and

$$m \geq n + \frac{|q'(0)| - |c||z_0|^n}{|q'(0)| + |c||z_0|^n} \tag{1.3}$$

2. Main results

In the beginning, we prove extension of Jake's Lemma [5] as follows:

Theorem 2.1. *Let $c = re^{it}$ with $-\frac{\pi\alpha}{\alpha+\lambda} < t < \frac{\pi\lambda}{\alpha+\lambda}$, where $0 < \alpha \leq 1$ and $0 < \lambda \leq 1$. Also let $0 \leq \beta \leq (\alpha+\lambda)|c|^{\frac{\alpha+\lambda}{2}}\cos(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})$ and $p \in \mathcal{H}_\beta[c^{\frac{\alpha+\lambda}{2}}, n]$ with $p(z) \neq 0$ in $\mathbb{U}$. If there exist elements $z_1 \in \mathbb{U}$ and $z_2 \in \mathbb{U}$ such that $|z_1| = |z_2| = r$ and for all $z \in \mathbb{U}_r = \{z \in \mathbb{C}, |z| < r\}$*

$$-\frac{\pi\alpha}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi\lambda}{2}, \tag{2.1}$$

then we have

$$z_1 p'(z_1) = -i\frac{\lambda+\alpha}{2}m_1 p(z_1), \tag{2.2}$$

and

$$z_2 p'(z_2) = i\frac{\lambda+\alpha}{2}m_2 p(z_2), \tag{2.3}$$

where

$$m_1 > \left(n + \frac{|c|^{\frac{\alpha+\lambda}{2}}\cos(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) - \frac{\beta}{\lambda+\alpha}}{|c|^{\frac{\alpha+\lambda}{2}}\cos(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + \frac{\beta}{\lambda+\alpha}}\right)\frac{1 + \sin(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}{\cos(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}, \tag{2.4}$$

and

$$m_2 > \left(n + \frac{|c|^{\frac{\alpha+\lambda}{2}}\cos(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) - \frac{\beta}{\lambda+\alpha}}{|c|^{\frac{\alpha+\lambda}{2}}\cos(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + \frac{\beta}{\lambda+\alpha}}\right)\frac{1 - \sin(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}{\cos(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}. \tag{2.5}$$

Proof. Let us define

$$q(z) = \exp\left\{\frac{\pi i(\lambda-\alpha)}{4}\right\}\left(\frac{c_1 + \bar{c}_1 z}{1-z}\right)^{\frac{\lambda+\alpha}{2}}$$

with $c_1 = c\exp\left\{\frac{-\pi i(\lambda-\alpha)}{2(\lambda+\alpha)}\right\}$. It is easy to find that q is analytic in $\mathbb{U}$, $q(0) = c^{\frac{\lambda+\alpha}{2}}$ and

$$-\frac{\pi\alpha}{2} < \arg q(\mathbb{U}) < \frac{\pi\lambda}{2},$$

moreover $q \in Q$ and $E(q) = 1$. Upon assumption and the properties of the function q, we have $p(z_1) \in q(\partial\mathbb{U})$ and $p(z_2) \in q(\partial\mathbb{U})$, also $p(\{z : |z| < r\}) \subset q(\mathbb{U})$. Define

$$p_1(z) = \exp\left\{\frac{-\pi i(\lambda-\alpha)}{2(\lambda+\alpha)}\right\}\{p(z)\}^{\frac{2}{\lambda+\alpha}} \qquad (z \in \mathbb{U}),$$

and

$$q_1(z) = \frac{c_1 + \bar{c_1} z}{1-z} \qquad (z \in \mathbb{U}),$$

with $c_1 = c\exp\left\{\frac{-\pi i(\lambda-\alpha)}{2(\lambda+\alpha)}\right\}$. Then it can be readily considered that $q_1 \in Q$, $q_1(0) = p_1(0)$, $q_1(\mathbb{U}) = \{w \in \mathbb{C} : \mathfrak{Re} w > 0\}$(note that $\mathfrak{Re} c_1 > 0$) and $p_1(\{z : |z| < r\}) \subset q_1(\mathbb{U})$. Also $p_1(z_1) = -ix_1$ and $p_1(z_2) = ix_2$, with $x_1, x_2 > 0$. By means of calculating the inverse of q_1 and obtaining the derivative of q_1, we reach to

$$q_1^{-1}(z) = \frac{z - c_1}{z + \bar{c_1}} \qquad and \qquad q_1'(z) = \frac{2\mathfrak{Re} c_1}{(1-z)^2}.$$

On the other hand, since $p \in \mathcal{H}_\beta[c^{\frac{\alpha+\lambda}{2}}, n]$, we have $p_1 \in \mathcal{H}_{c_2}[a, n]$, with

$$a = c\exp\left\{\frac{\pi i(\alpha-\lambda)}{2(\lambda+\alpha)}\right\} = c_1 \qquad and \qquad c_2 = \frac{2c^{\frac{2-\alpha-\lambda}{2}}\beta}{\alpha+\lambda}\exp\left\{\frac{\pi i(\alpha-\lambda)}{2(\lambda+\alpha)}\right\}.$$

Hence by applying Lemma 1.1 we deduce that there exist complex numbers ζ_1 and ζ_2 in $\partial\mathbb{U}$ such that $p_1(z_1) = q_1(\zeta_1)$ and $p_1(z_2) = q_1(\zeta_2)$ and also

$$z_1 p_1'(z_1) = k_1\zeta_1 q_1'(\zeta_1) \qquad and \qquad z_2 p_1'(z_2) = k_2\zeta_2 q_1'(\zeta_2),$$

where

$$k_1 \geq n + \frac{|q_1'(0)| - |c_2||z_1|^n}{|q_1'(0)| + |c_2||z_1|^n} \qquad and \qquad k_2 \geq n + \frac{|q_1'(0)| - |c_2||z_2|^n}{|q_1'(0)| + |c_2||z_2|^n}.$$

Since $p_1(z_1) = -ix_1$ with $x_1 > 0$ and $\zeta_1 = q_1^{-1}(p_1(z_1)) = \frac{ix_1+c_1}{ix_1-\bar{c_1}}$, we have

$$\begin{aligned}
\frac{z_1 p'(z_1)}{p(z_1)} &= \frac{\lambda+\alpha}{2}\frac{z_1 p_1'(z_1)}{p_1(z_1)} \\
&= \frac{\lambda+\alpha}{2}\frac{k_1\zeta_1 q_1'\zeta_1)}{p_1(z_1)} \\
&= k_1\frac{\lambda+\alpha}{2}\frac{ix_1+c_1}{ix_1-\bar{c_1}} \times \frac{1}{-ix_1} \times \frac{2\mathfrak{Re} c_1}{(1-\frac{ix_1+c_1}{ix_1-\bar{c_1}})^2} \\
&= k_1\frac{\lambda+\alpha}{2}\frac{1}{ix_1} \times \frac{x_1^2 + 2x_1\mathfrak{Im} c_1 + |c_1|^2}{2\mathfrak{Re} c_1} \\
&= -ik_1\left(\frac{\lambda+\alpha}{2}\right)\frac{x_1^2 + 2|c|x_1\sin(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_1\cos(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}.
\end{aligned}$$

Set

$$f(x) = \frac{x^2 + 2|c|x\sin(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x\cos(t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})} \qquad (x > 0).$$

By computing, it can be easily observed that

$$\min_{x>0} f(x) = f(|c|) = \frac{1+\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}{\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}.$$

Now using $q_1'(0) = 2|c|\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})$ and $|c_2| = \frac{2\beta|c|^{\frac{2-\alpha-\lambda}{2}}}{\lambda+\alpha}$, we obtain

$$m_1 = k_1 f(x_1) > \left(n + \frac{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) - \frac{\beta}{\lambda+\alpha}}{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + \frac{\beta}{\lambda+\alpha}}\right)\frac{1+\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}{\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}$$

Thus assertions (2.2) and (2.4) hold. Now similar to the procedure of the former case, since $p_1(z_2) = ix_2$, with $x_2 > 0$ and $\zeta_2 = q_1^{-1}(ix_2) = \frac{ix_2-c_1}{ix_2+\bar{c_1}}$ we can obtain

$$\begin{aligned}
\frac{z_2p'(z_2)}{p(z_2)} &= \frac{\lambda+\alpha}{2}\frac{z_2p_1'(z_2)}{p_1(z_2)} \\
&= \frac{\lambda+\alpha}{2}\frac{k_2\zeta_2 q_1'\zeta_2)}{p_1(z_2)} \\
&= k_2\frac{\lambda+\alpha}{2}\frac{ix_2-c_1}{ix_2+\bar{c_1}} \times \frac{1}{ix_2} \times \frac{2\mathfrak{Re}c_1}{(1-\frac{ix_2-c_1}{ix_2+\bar{c_1}})^2} \\
&= k_2\frac{\lambda+\alpha}{2}\frac{1}{ix_2} \times \frac{-x_2^2+2x_2\mathfrak{Im}c_1-|c_1|^2}{2\mathfrak{Re}c_1} \\
&= ik_2\left(\frac{\lambda+\alpha}{2}\right)\frac{x_2^2-2|c|x_2\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})+|c|^2}{2|c|x_2\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}.
\end{aligned}$$

Set

$$g(x) = \frac{x^2-2|c|x\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})+|c|^2}{2|c|x\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})} \qquad (x>0).$$

By computing, we have

$$\min_{x>0} g(x) = g(|c|) = \frac{1-\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}{\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}.$$

Thus in view of $q_1'(0) = 2|c|\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})$ and $|c_2| = \frac{2\beta|c|^{\frac{2-\alpha-\lambda}{2}}}{\lambda+\alpha}$, as the former case, we can conclude assertions (2.3) and (2.5). □

Remark 2.2. Note that the above theorem extends Theorem 2.1 obtained in [3].

By applying the same trend of Theorem 2.1 and putting $\alpha = \lambda$ in this theorem, we obtain

Corollary 2.3. *Let $c = re^{it}$ be a complex number with $\mathfrak{Re}c > 0$. Let $0 \leq \beta \leq 2\lambda|c|^\lambda\cos t$ and $p \in \mathcal{H}_\beta[c^\lambda, n]$ with $p(z) \neq 0$ in $\mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg p(z)| < \frac{\lambda\pi}{2} \quad \textit{for} \;\; |z| < |z_0|,$$

and $p(z_0)^{\frac{1}{\lambda}} = \pm ia$, *where* $a > 0$ *and* $0 < \lambda \leq 1$, *Then we have*

$$z_0 p'(z_0) = im\lambda p(z_0),$$

where

$$m > \frac{a^2 - 2a|c|sint + |c|^2}{2a|c|\cos t}\left(n + \frac{|c|^\lambda \cos t - \frac{\beta}{2\lambda}}{|c|^\lambda \cos t + \frac{\beta}{2\lambda}}\right) \quad when \quad \arg p(z_0) = \frac{\lambda\pi}{2},$$

and

$$m < -\frac{a^2 + 2a|c|sint + |c|^2}{2a|c|\cos t}\left(n + \frac{|c|^\lambda \cos t - \frac{\beta}{2\lambda}}{|c|^\lambda \cos t + \frac{\beta}{2\lambda}}\right) \quad when \quad \arg p(z_0) = \frac{-\lambda\pi}{2}.$$

By putting $\lambda = 1$ in Corollary 2.1, we have

Corollary 2.4. *Let* $c = re^{it}$ *be a complex number with* $\mathfrak{Re} c > 0$. *Let* $0 \leq \beta \leq 2\mathfrak{Re} c$ *and* $p \in \mathcal{H}_\beta[c, n]$. *If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$|\arg p(z)| < \frac{\pi}{2} \quad for \ |z| < |z_0|,$$

and $p(z_0) = \pm ia$ *where* $a > 0$, *Then we have*

$$z_0 p'(z_0) = imp(z_0),$$

where

$$m > \frac{a^2 - 2a\mathfrak{Im} p(0) + |p(0)|^2}{2a\mathfrak{Re} p(0)}\left(n + \frac{2\mathfrak{Re} p(0) - \beta}{2\mathfrak{Re} p(0) + \beta}\right) \quad when \quad \arg p(z_0) = \frac{\pi}{2},$$

and

$$m < -\frac{a^2 + 2a\mathfrak{Im} p(0) + |p(0)|^2}{2a\mathfrak{Re} p(0)}\left(n + \frac{2\mathfrak{Re} p(0) - \beta}{2\mathfrak{Re} p(0) + \beta}\right) \quad when \quad \arg p(z_0) = -\frac{\pi}{2}.$$

Remark 2.5. Letting $p \in \mathcal{H}[c, 1]$ in corollary 2.2 and using the corrections needed in this Corollary, one can gain Theorem 2.1 in [11].

By setting $c = 1$ in Corollary 2.2, we attain

Corollary 2.6. *Let* $p \in \mathcal{H}_\beta[1, n]$ *and* $0 \leq \beta \leq 2$. *If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$|\arg p(z)| < \frac{\pi}{2} \quad for \ |z| < |z_0|,$$

and $p(z_0) = \pm ia$ *where* $a > 0$, *Then we have*

$$z_0 p'(z_0) = imp(z_0),$$

where

$$m > \frac{1}{2}(a + a^{-1})\left(n + \frac{2 - \beta}{2 + \beta}\right) \quad when \quad \arg p(z_0) = \frac{\pi}{2},$$

and

$$m < -\frac{1}{2}(a + a^{-1})\left(n + \frac{2 - \beta}{2 + \beta}\right) \quad when \quad \arg p(z_0) = -\frac{\pi}{2}.$$

Remark 2.7. Letting $p \in \mathcal{H}[1, 1]$ in Corollary 2.3 and implying the alternations required in this corollary, we can obtain Theorem 1 in [10].

Theorem 2.8. *(extension of open door Lemma) Let* $c = re^{it}$ *with* $-\frac{\pi\alpha}{\alpha+\lambda} < t < \frac{\pi\lambda}{\alpha+\lambda}$, *where* $0 < \alpha \le 1$ *and* $0 < \lambda \le 1$. *Also let* $0 \le \beta \le (\alpha+\lambda)|c|^{\frac{\alpha+\lambda}{2}} \cos B$ *and* $p \in \mathcal{H}_\beta[c^{\frac{\alpha+\lambda}{2}}, n]$ *with* $p(z) \ne 0$ *in* $\mathbb{U}$. *If*

$$\gamma p(z)^{\frac{2}{\alpha+\lambda}} + \frac{2}{\alpha+\lambda}\frac{zp'(z)}{p(z)} \ne iy \qquad (z \in \mathbb{U}),$$

for all $y \in \mathbb{R}$ *where*

$$y > \frac{\sqrt{M}}{cosB}(\sqrt{M + 2|c|\cos B} - \sqrt{M}\sin B),$$

or

$$y < -\frac{\sqrt{M}}{cosB}(\sqrt{M + 2|c|\cos B} + \sqrt{M}\sin B),$$

then

$$-\frac{\alpha\pi}{2} < \arg p(z) < \frac{\lambda\pi}{2} \qquad (z \in \mathbb{U}), \tag{2.6}$$

where $\gamma = \exp\{-i\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}\}$, $B = t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}$ *and* $M = n + \frac{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) - \frac{\beta}{\lambda+\alpha}}{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + \frac{\beta}{\lambda+\alpha}}$.

Proof. Let us set

$$p_1(z) = \exp\left\{\frac{-\pi i(\lambda-\alpha)}{2(\lambda+\alpha)}\right\}\{p(z)\}^{\frac{2}{\lambda+\alpha}} \qquad (z \in \mathbb{U}),$$

and

$$q_1(z) = \frac{c_1 + \bar{c_1}z}{1-z} \qquad (z \in \mathbb{U}),$$

where $c_1 = c\exp\left\{\frac{-\pi i(\lambda-\alpha)}{2(\lambda+\alpha)}\right\}$. We know that $p_1 \in \mathcal{H}_{c_2}[a, n]$, with

$$a = c\exp\left\{\frac{\pi i(\alpha-\lambda)}{2(\lambda+\alpha)}\right\} = c_1 \qquad and \qquad c_2 = \frac{2c^{\frac{2-\alpha-\lambda}{2}}\beta}{\alpha+\lambda}\exp\left\{\frac{\pi i(\alpha-\lambda)}{2(\lambda+\alpha)}\right\}.$$

and $p_1(0) = q_1(0)$. If $p(\mathbb{U})$ is not contained in the sector $\{w : -\frac{\pi\alpha}{2} < \arg w < \frac{\pi\lambda}{2}\}$, then $p_1\mathbb{U})$ is not contained in the right half plane $\mathfrak{Re}\,w > 0$. On the other hand $q_1(\mathbb{U}) = \{w : \mathfrak{Re}\,w > 0\}$, thus we follow that $p_1 \not\prec q_1$, then there exists a point $z_1 \in \mathbb{U}$ such that $p_1(\{z : |z| < |z_1|\}) \subset q_1(\mathbb{U})$ and $p_1(z_1) = -ix_1$ or $p_1(z_1) = ix_2$ with $x_1, x_2 > 0$. Let $p_1(z_1) = -ix_1$, with $x_1 > 0$. Similar to the argument of Theorem 2.1 we have

$$\frac{z_1p'(z_1)}{p(z_1)} = -ik_1\left(\frac{\lambda+\alpha}{2}\right)\frac{x_1^2 + 2|c|x_1\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_1\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})},$$

where $k_1 > M$. Then it yields

$$\begin{aligned}
&\mathfrak{Im}\left\{\gamma p(z_1)^{\frac{2}{\alpha+\lambda}} + \frac{2}{\alpha+\lambda}\frac{zp'(z_1)}{p(z_1)}\right\}\\
&= \mathfrak{Im}\left\{-ix_1 - ik_1\frac{x_1^2 + 2|c|x_1\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_1\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}\right\}\\
&= -(x_1 + k_1\frac{x_1^2 + 2|c|x_1\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_1\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})})\\
&< -(x_1 + M\frac{x_1^2 + 2|c|x_1\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_1\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})})
\end{aligned}$$

Suppose

$$f(x) = x + M\frac{x^2 + 2|c|x\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})} \qquad (x > 0).$$

By computing, we can readily find that

$$\min_{x>0} f(x) = f\left(\frac{|c|\sqrt{M}}{\sqrt{M + 2|c|\cos B}}\right) = \frac{\sqrt{M}}{\cos B}\left(\sqrt{M + 2|c|\cos B} + \sqrt{M}\sin B\right),$$

this implies that

$$\mathfrak{Im}\left\{\gamma p(z_1)^{\frac{2}{\alpha+\lambda}} + \frac{2}{\alpha+\lambda}\frac{z_1p'(z_1)}{p(z_1)}\right\} < -\frac{\sqrt{M}}{cosB}(\sqrt{M + 2|c|\cos B} + \sqrt{M}\sin B),$$

where $\gamma = \exp\{-i\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}\}$, $B = t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}$ and $M = n + \frac{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) - \frac{\beta}{\lambda+\alpha}}{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + \frac{\beta}{\lambda+\alpha}}$. On the other hand we have

$$\mathfrak{Re}\left\{\gamma p(z_1)^{\frac{2}{\alpha+\lambda}} + \frac{2}{\alpha+\lambda}\frac{z_1p'(z_1)}{p(z_1)}\right\} = 0,$$

that this contradicts with the hypothesis. For the case $p_1(z_1) = ix_2$, Similar to the argument of Theorem 2.1 we have

$$\frac{z_1p'(z_1)}{p(z_1)} = ik_2\left(\frac{\lambda+\alpha}{2}\right)\frac{x_2^2 - 2|c|x_2\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_2\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})},$$

where $k_2 > M$. Then it yields

$$\begin{aligned}
&\mathfrak{Im}\left\{\gamma p(z_1)^{\frac{2}{\alpha+\lambda}} + \frac{2}{\alpha+\lambda}\frac{zp'(z_1)}{p(z_1)}\right\}\\
&= \mathfrak{Im}\left\{ix_2 + ik_2\frac{x_2^2 - 2|c|x_2\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_2\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}\right\}\\
&= x_2 + k_2\frac{x_2^2 - 2|c|x_2\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_2\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}\\
&> x_2 + M\frac{x_2^2 - 2|c|x_2\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x_2\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})}
\end{aligned}$$

Suppose

$$g(x) = x + M\frac{x^2 - 2|c|x\sin(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + |c|^2}{2|c|x\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)})} \qquad (x > 0).$$

By computing we can easily conclude that

$$\min_{x>0} g(x) = g\left(\frac{|c|\sqrt{M}}{\sqrt{M+2|c|\cos B}}\right) = \frac{\sqrt{M}}{\cos B}\left(\sqrt{M+2|c|\cos B} - \sqrt{M}\sin B\right),$$

thus we have

$$\mathfrak{Im}\left\{\gamma p(z_1)^{\frac{2}{\alpha+\lambda}} + \frac{2}{\alpha+\lambda}\frac{zp'(z_1)}{p(z_1)}\right\} > \frac{\sqrt{M}}{cosB}(\sqrt{M+2|c|\cos B} - \sqrt{M}\sin B),$$

where $\gamma = \exp\{-i\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}\}$, $B = t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}$ and $M = n + \frac{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) - \frac{\beta}{\lambda+\alpha}}{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + \frac{\beta}{\lambda+\alpha}}$.
On the other hand we have

$$\mathfrak{Re}\left\{\gamma p(z_1)^{\frac{2}{\alpha+\lambda}} + \frac{2}{\alpha+\lambda}\frac{z_1p'(z_1)}{p(z_1)}\right\} = 0,$$

that this contradicts with the hypothesis. Hence the assertion (2.6) holds. □

Remark 2.9. we note that Theorem 2.2 extends Theorem 2.1 in [4]

Also we can write the other version of extension of open door Lemma as follows:

Corollary 2.10. *Let $c = re^{it}$ be a complex number with $\mathfrak{Re}c > 0$. Also Let $0 < \lambda \le 1$, $0 \le \beta \le 2\lambda|c|^\lambda\cos t$ and $p \in \mathcal{H}_\beta[c^\lambda, n]$ with $p(z) \neq 0$ in $\mathbb{U}$. If*

$$p(z)^{\frac{1}{\lambda}} + \frac{1}{\lambda}\frac{zp'(z)}{p(z)} \neq iy \qquad (z \in \mathbb{U}),$$

for all $y \in \mathbb{R}$, where

$$y > \frac{\sqrt{M}}{\cos t}\left(\sqrt{M+2|c|\cos t} - \sqrt{M}\sin t\right),$$

or

$$y < -\frac{\sqrt{M}}{\cos t}\left(\sqrt{M+2|c|\cos t} + \sqrt{M}\sin t\right),$$

then

$$-\frac{\lambda\pi}{2} < \arg p(z) < \frac{\lambda\pi}{2} \qquad (z \in \mathbb{U}),$$

where $M = n + \frac{|c|^\lambda \cos t - \frac{\beta}{2\lambda}}{|c|^\lambda \cos t + \frac{\beta}{2\lambda}}$.

Proof. The proof of this corollary is similar to that of Theorem 2.2 (put $\alpha = \lambda$), so we omit its details. □

Corollary 2.11. *Let* $f \in \mathcal{A}_{n,b}$ *with* $f(z)f'(z) \neq 0$ *in* $\mathbb{U} - \{0\}$. *Also let* $\alpha + \lambda = \frac{2}{t_1}$ *with* $t_1 \geq 1$ *and* $0 \leq b \leq \frac{2}{n} \cos\{-\frac{\pi t_1(\lambda-\alpha)}{4}\}$. *If*

$$(\gamma - 1)\frac{zf'(z)}{f(z)} + (1 + \frac{zf''(z)}{f'(z)}) \neq iy \qquad (z \in \mathbb{U}),$$

for all $y \in \mathbb{R}$ *where*

$$y > \frac{\sqrt{M}}{\cos\left\{-\frac{\pi t_1(\lambda-\alpha)}{4}\right\}} \left(\sqrt{M + \cos\left\{-\frac{\pi t_1(\lambda-\alpha)}{4}\right\}} - \sqrt{M} \sin\left\{-\frac{\pi t_1(\lambda-\alpha)}{4}\right\}\right),$$

or

$$y < -\frac{\sqrt{M}}{\cos\left\{-\frac{\pi t_1(\lambda-\alpha)}{4}\right\}} \left(\sqrt{M + \cos\left\{-\frac{\pi t_1(\lambda-\alpha)}{4}\right\}} + \sqrt{M} \sin\left\{-\frac{\pi t_1(\lambda-\alpha)}{4}\right\}\right),$$

then

$$-\frac{\pi}{2}\alpha t_1 < \arg \frac{zf'(z)}{f(z)} < \frac{\pi}{2}\lambda t_1 \qquad (z \in \mathbb{U}),$$

where $\gamma = \exp(-i\pi \frac{t_1(\lambda-\alpha)}{4})$ *and* $M = n + \frac{\cos\left\{-\frac{\pi t_1(\lambda-\alpha)}{4}\right\} - \frac{nb}{2}}{\cos\left\{-\frac{\pi t_1(\lambda-\alpha)}{4}\right\} + \frac{nb}{2}}$.

Proof. Let $p(z) = (\frac{zf'(z)}{f(z)})^{\frac{1}{t_1}}$, then we have $p \in \mathcal{H}_{\frac{nb}{t_1}}[1, n]$ with $p(z) \neq 0$ in $\mathbb{U}$. Then with applying Theorem 2.2 and with letting $c = 1$, $t = 0$, $\alpha + \lambda = \frac{2}{t_1}$ and $\beta = \frac{nb}{t_1}$ in this theorem, the proof is complete. □

Theorem 2.12. *Let* $c = re^{it}$ *with* $-\frac{\pi\alpha}{\alpha+\lambda} < t < \frac{\pi\lambda}{\alpha+\lambda}$, *where* $0 < \alpha \leq 1$ *and* $0 < \lambda \leq 1$. *Also let* $M > \frac{2|c|}{\cos B}$, $0 \leq \beta \leq (\alpha + \lambda)|c|^{\frac{\alpha+\lambda}{2}} \cos B$ *and* $p \in \mathcal{H}_\beta[c^{\frac{\alpha+\lambda}{2}}, n]$ *with* $p(z) \neq 0$ *in* $\mathbb{U}$. *If*

$$\gamma p(z)^{\frac{2}{\alpha+\lambda}} - \frac{2}{\alpha + \lambda} \frac{zp'(z)}{p(z)} \neq iy \qquad (z \in \mathbb{U}),$$

for all $y \in \mathbb{R}$ *where*

$$y > \frac{\sqrt{M}}{cosB}(\sqrt{M - 2|c| \cos B} + \sqrt{M} \sin B),$$

or

$$y < -\frac{\sqrt{M}}{cosB}(\sqrt{M - 2|c| \cos B} - \sqrt{M} \sin B),$$

then

$$-\frac{\alpha\pi}{2} < \arg p(z) < \frac{\lambda\pi}{2} \qquad (z \in \mathbb{U}),$$

where $\gamma = \exp\{-i\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}\}$, $B = t - \pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}$ *and* $M = n + \frac{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) - \frac{\beta}{\lambda+\alpha}}{|c|^{\frac{\alpha+\lambda}{2}}\cos(t-\pi\frac{\lambda-\alpha}{2(\lambda+\alpha)}) + \frac{\beta}{\lambda+\alpha}}$.

Proof. The proof of this theorem is similar to Theorem 2.2, and we omit its details. □

Corollary 2.13. *Let* $f(z) = \frac{1}{z} + \beta z^n + \dots$ *be a meromorphic function with* $f'f \neq 0$ *in* $\mathbb{U} - \{0\}$. *Also let* $-\frac{2}{(n+1)} \leq \beta \leq 0$ *and* $M > 2$. *If*

$$-1 - \frac{zf''(z)}{f'(z)} \neq iy \qquad (z \in \mathbb{U}),$$

for all $y \in \mathbb{R}$ *where*

$$y > \sqrt{M}(\sqrt{M-2}),$$

or

$$y < -\sqrt{M}(\sqrt{M-2}),$$

then we have

$$-\frac{\pi}{2} < \arg\left\{-\frac{zf'(z)}{f(z)}\right\} < \frac{\pi}{2} \qquad (z \in \mathbb{U}),$$

where $M = (n+1) + \frac{2+(n+1)\beta}{2-(n+1)\beta}$.

Proof. Let $p(z) = -\frac{zf'(z)}{f(z)}$, then $p \in \mathcal{H}_{\beta_1}[1, n+1]$ with $\beta_1 = -(n+1)\beta > 0$. With a simple computation we obtain

$$p(z) - \frac{zp'(z)}{p(z)} = -1 - \frac{zf''(z)}{f'(z)} \qquad (z \in \mathbb{U}).$$

Then with using Theorem 2.3 and with letting $c = 1$, $t = 0$, $\alpha = \lambda = 1$ and also with substituting β by β_1 in this theorem, we obtain this result and the proof is complete. □

3. Further applications related to extension of Jake's Lemma

Corollary 3.1. *Let* $0 < \lambda \leq 1$, $c \in \mathbb{C}$ *and* β_1 *be a real number such that* $(c^\lambda - \beta_1)^{\frac{1}{\lambda}} = re^{it}$ *with* $\mathfrak{Re}(c^\lambda - \beta_1)^{\frac{1}{\lambda}} > 0$. *Suppose* $0 \leq \beta \leq 2\lambda|c^\lambda - \beta_1|\cos t$ *and* $p \in \mathcal{H}_\beta[c^\lambda, n]$ *with* $p(z) \neq \beta_1$ *in* $\mathbb{U}$. *If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$|\arg(p(z) - \beta_1)| < \frac{\lambda\pi}{2} \qquad for \ \ |z| < |z_0|,$$

and $(p(z_0) - \beta_1)^{\frac{1}{\lambda}} = \pm ia$, *where* $a > 0$, *Then we have*

$$\frac{z_0 p'(z_0)}{p(z_0) - \beta_1} = im\lambda,$$

where for $\arg\{p(z_0) - \beta_1\} = \frac{\lambda\pi}{2}$

$$m > \frac{a^2 - 2a\mathfrak{Im}(c^\lambda - \beta_1)^{\frac{1}{\lambda}} + |c^\lambda - \beta_1|^{\frac{2}{\lambda}}}{2a\mathfrak{Re}(c^\lambda - \beta_1)^{\frac{1}{\lambda}}}\left(n + \frac{|c^\lambda - \beta_1|\cos t - \frac{\beta}{2\lambda}}{|c^\lambda - \beta_1|\cos t + \frac{\beta}{2\lambda}}\right),$$

and for $\arg\{p(z_0)-\beta_1\} = -\frac{\lambda\pi}{2}$

$$m < -\frac{a^2 + 2a\mathfrak{Im}(c^\lambda - \beta_1)^{\frac{1}{\lambda}} + |c^\lambda - \beta_1|^{\frac{2}{\lambda}}}{2a\mathfrak{Re}(c^\lambda - \beta_1)^{\frac{1}{\lambda}}}\left(n + \frac{|c^\lambda - \beta_1|\cos t - \frac{\beta}{2\lambda}}{|c^\lambda - \beta_1|\cos t + \frac{\beta}{2\lambda}}\right).$$

Proof. It is sufficient that we consider $q(z) = p(z) - \beta_1$. Then $q(z) \in \mathcal{H}_\beta[c_1^\lambda, n]$ with $c_1 = (c^\lambda - \beta_1)^{\frac{1}{\lambda}}$. Also from the hypothesis we have $\mathfrak{Re}c_1 > 0$ and there exists a point $z_0 \in \mathbb{U}$ such that $|\arg q(z)| < \frac{\lambda\pi}{2}$ for $|z| < |z_0|$ and $q(z_0)^{\frac{1}{\lambda}} = \pm ia$. Now using Corollary 2.1 for q, we get the result and the proof is complete. □

By using Corollary 3.1, we obtain

Corollary 3.2. *Let* $f \in \mathcal{A}_{n,b}$ *with* $\frac{f(z)}{z} \neq \beta$ *in* $\mathbb{U}$. *Suppose* $0 \leq \beta < 1$ *and* $0 \leq b \leq 2(1-\beta)$. *If*

$$\frac{zf'(z) - f(z)}{f(z) - \beta z} \neq is \qquad (z \in \mathbb{U}),$$

for all $s \in \mathbb{R}$ *where* $|s| > n + \frac{2(1-\beta)-b}{2(1-\beta)+b}$, *then we have* $\mathfrak{Re}\frac{f(z)}{z} > \beta$.

Proof. Let us define $p(z) = \frac{f(z)}{z}$, then $p \in \mathcal{H}_b[1,n]$. Let there exists a point $z_0 \in \mathbb{U}$ such that $\mathfrak{Re}p(z) > \beta$ for $|z| < |z_0|$ and $\mathfrak{Re}p(z_0) = \beta$, so $|\arg(p(z) - \beta)| < \frac{\pi}{2}$ for $|z| < |z_0|$ and $p(z_0) = \beta \pm ia$, where $a > 0$. Now applying Corollary 3.1, we have

$$\frac{z_0 f'(z_0) - f(z_0)}{f(z_0) - \beta z_0} = \frac{z_0 p'(z_0)}{p(z_0) - \beta} = im \qquad (z \in \mathbb{U}),$$

where for $p(z_0) - \beta = ia$

$$m > \frac{a^2 - (1-\beta)^2}{2a(1-\beta)}\left(n + \frac{2(1-\beta)-b}{2(1-\beta)+b}\right) \geq \left(n + \frac{2(1-\beta)-b}{2(1-\beta)+b}\right),$$

and for $p(z_0) - \beta = -ia$

$$m < -\frac{a^2 - (1-\beta)^2}{2a(1-\beta)}\left(n + \frac{2(1-\beta)-b}{2(1-\beta)+b}\right) \leq -\left(n + \frac{2(1-\beta)-b}{2(1-\beta)+b}\right),$$

which contradicts with the hypothesis. Hence the proof is complete. □

Also similar to Corollary 3.1, we can conclude

Corollary 3.3. *Let* $0 < \lambda \leq 1$, $c \in \mathbb{C}$ *and* β_1 *be a real number such that* $(\beta_1 - c)^{\frac{1}{\lambda}} = re^{it}$ *with* $\mathfrak{Re}(\beta_1 - c)^{\frac{1}{\lambda}} > 0$. *Suppose* $-2\lambda|\beta_1 - c|\cos t \leq \beta \leq 0$ *and* $p \in \mathcal{H}_\beta[c,n]$ *with* $p(z) \neq \beta_1$ *in* $\mathbb{U}$. *If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$|\arg(\beta_1 - p(z))| < \frac{\lambda\pi}{2} \quad \textit{for} \;\; |z| < |z_0|,$$

and $(\beta_1 - p(z_0))^{\frac{1}{\lambda}} = \pm ia$, *where* $a > 0$, *Then we have*

$$\frac{z_0 p'(z_0)}{p(z_0) - \beta_1} = im\lambda,$$

where for $\arg\{\beta_1 - p(z_0)\} = \frac{\lambda\pi}{2}$

$$m > \frac{a^2 - 2a\mathfrak{Im}(\beta_1 - c)^{\frac{1}{\lambda}} + |\beta_1 - c|^{\frac{2}{\lambda}}}{2a\mathfrak{Re}(\beta_1 - c)^{\frac{1}{\lambda}}}\left(n + \frac{|\beta_1 - c|\cos t + \frac{\beta}{2\lambda}}{|\beta_1 - c|\cos t - \frac{\beta}{2\lambda}}\right)$$

and for $\arg\{\beta_1 - p(z_0)\} = -\frac{\lambda\pi}{2}$

$$m < -\frac{a^2 + 2a\mathfrak{Im}(\beta_1 - c)^{\frac{1}{\lambda}} + |\beta_1 - c|^{\frac{2}{\lambda}}}{2a\mathfrak{Re}(\beta_1 - c)^{\frac{1}{\lambda}}}\left(n + \frac{|\beta_1 - c|\cos t + \frac{\beta}{2\lambda}}{|\beta_1 - c|\cos t - \frac{\beta}{2\lambda}}\right).$$

Proof. It is sufficient to consider $q(z) = \beta_1 - p(z)$. The rest of the proof is similar to the proof of Corollary 3.1. □

The same as Corollary 3.2 and by applying Corollary 3.3, we can obtain the following Corollary.

Corollary 3.4. *Let* $\beta > 1$ *and* $-2(\beta - 1) \leq b \leq 0$*. Suppose* $f \in \mathcal{A}_{n,b}$ *with* $\frac{f(z)}{z} \neq \beta$*. in* $\mathbb{U}$*. If*

$$\frac{zf'(z) - f(z)}{f(z) - \beta z} \neq is \qquad (z \in \mathbb{U}),$$

for all $s \in \mathbb{R}$ *where* $|s| > n + \frac{2(\beta-1)+b}{2(\beta-1)-b}$*, then we have*

$$\mathfrak{Re}\frac{f(z)}{z} < \beta.$$

Theorem 3.5. *Let* c*,* β_1 *and* γ *be real numbers with* $c^\alpha - \beta_1 > 0$*. Suppose* $\gamma > 0$*,* $0 < \alpha \leq 1$*,* $0 \leq \beta_1 < 1$ *and* $0 \leq \beta \leq 2\alpha(c^\alpha - \beta_1)$*. If* $p \in \mathcal{H}_\beta[c^\alpha, n]$ *with* $p(z) \neq \beta_1$ *in* $\mathbb{U}$ *and*

$$|\arg(p(z) - \beta_1 + \gamma zp'(z))| \leq \frac{\pi}{2}(\alpha + \frac{2}{\pi}\tan^{-1}(\alpha\gamma s)) \qquad (z \in \mathbb{U}),$$

then

$$|\arg(p(z) - \beta_1)| < \frac{\pi}{2}\alpha \qquad \in \mathbb{U},$$

where $s = n + \left(\frac{(c^\alpha - \beta_1) - \frac{\beta}{2\alpha}}{(c^\alpha - \beta_1) + \frac{\beta}{2\alpha}}\right)$.

Proof. If there exists a point $z_0 \in \mathbb{U}$ such that $|\arg(p(z) - \beta_1)| < \frac{\pi}{2}\alpha$ for $|z| < |z_0|$ and $|\arg(p(z_0) - \beta_1)| = \frac{\pi}{2}\alpha$, then from Corollary 3.1 we have

$$\frac{z_0 p'(z_0)}{p(z_0) - \beta_1} = i\alpha m,$$

where

$$|m| > \left(n + \frac{(c^\alpha - \beta_1) - \frac{\beta}{2\alpha}}{(c^\alpha - \beta_1) + \frac{\beta}{2\alpha}}\right) = s.$$

Thus for the case $\arg(p(z_0)-\beta_1)=\frac{\pi}{2}\alpha$ we have

$$\begin{aligned}\arg\{p(z_0)-\beta_1+\gamma z_0p'(z_0)\} &= \arg\left\{(p(z_0)-\beta_1)(1+\gamma\frac{z_0p'(z_0)}{p(z_0)-\beta_1})\right\}\\ &= \frac{\pi}{2}\alpha+\arg\{1+i\gamma\alpha m\}\\ &> \frac{\pi}{2}\alpha+\tan^{-1}(\gamma\alpha s)\end{aligned}$$

which contradicts with the hypothesis. Also for the case $\arg(p(z_0)-\beta_1)=-\frac{\pi}{2}\alpha$ we have

$$\begin{aligned}\arg\{p(z_0)-\beta_1+\gamma z_0p'(z_0)\} &= \arg\left\{(p(z_0)-\beta_1)(1+\gamma\frac{z_0p'(z_0)}{p(z_0)-\beta_1})\right\}\\ &= -\frac{\pi}{2}\alpha+\arg\{1+i\gamma\alpha m\}\\ &< -(\frac{\pi}{2}\alpha+\tan^{-1}(\gamma\alpha s))\end{aligned}$$

which contradicts with the hypothesis. Hence the proof is complete. □

By putting $c=\gamma=\alpha=n=1$ in Theorem 3.1 we have

Corollary 3.6. *Let $0\le\beta_1<1$ be a real number and $0\le\beta\le 2(1-\beta_1)$. If $p(z)=1+\beta z+\dots$ with $p(z)\neq\beta_1$ in $\mathbb{U}$ and*

$$|\arg(p(z)-\beta_1+zp'(z))|\le\frac{\pi}{2}+\tan^{-1}\left\{\frac{4-4\beta_1}{(2-2\beta_1)+\beta}\right\}\qquad(z\in\mathbb{U}),$$

then

$$\mathfrak{Re}\,p(z)>\beta_1\qquad z\in\mathbb{U}.$$

Remark 3.7. Letting $p\in\mathcal{H}[1,1]$ in the Corollary 3.5 and applying the reforms required in this corollary, we can obtain Theorem 3 in [13].

Theorem 3.8. *Let $-\lambda<b<\lambda$, $\lambda>0$ and $k>0$. Also let $p\in\mathcal{H}_\beta[1,n]$ with $p(z)\neq\frac{2\lambda}{b+\lambda}$ in $\mathbb{U}$ and $0\le\beta\le 1-\frac{b}{\lambda}$. If for all $z\in\mathbb{U}$*

$$\mathfrak{Re}\left\{p(z)+k\frac{zp'(z)}{p(z)}\right\}\le\begin{cases}Mk\frac{\lambda+b}{2(\lambda-b)} & \text{if}\quad -\lambda<b\le 0, M\ge\frac{2(\lambda-b)}{k(\lambda+b)}\\ \frac{Mk}{2}\frac{\lambda-b}{\lambda+b}+\frac{2\lambda}{\lambda+b} & \text{if}\quad \frac{\lambda}{1+kM}\le b<\lambda\\ \frac{Mk}{2}\frac{\lambda-b}{\lambda+b} & \text{if}\quad 0<b<\frac{\lambda}{1+kM}, M\ge\frac{2(\lambda+b)}{k(\lambda-b)},\end{cases}$$

then we have

$$\left|p(z)-\frac{\lambda}{b+\lambda}\right|<\frac{\lambda}{b+\lambda}\qquad z\in\mathbb{U},$$

where $M=n+\frac{\frac{\lambda-b}{\lambda}-\beta}{\frac{\lambda-b}{\lambda}+\beta}$.

Proof. Let us define

$$q(z)=\frac{\lambda(1-z)}{\lambda-bz}.$$

One can easily observe that $q \in Q$ with $q(0) = p(0) = 1$ and q maps the open unit disc $\mathbb{U}$ onto the disk with the center $\frac{\lambda}{\lambda+b}$ and the radius $\frac{\lambda}{\lambda+b}$. Moreover

$$q^{-1}(z) = \frac{\lambda(z-1)}{bz-\lambda} \qquad \textit{and} \qquad q'(z) = \frac{\lambda(b-\lambda)}{(\lambda-bz)^2}.$$

We claim that $p \prec q$, otherwise if $p \not\prec q$, then there exist points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that $p(z_0) = q(\zeta_0)$ and $p(\{z : |z| < |z_0|\}) \subset q(\mathbb{U})$. Therefore from lemma 1.1 we have

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0).$$

where

$$m \geq n + \frac{|q'(0)| - |\beta||z_0|^n}{|q'(0)| + |\beta||z_0|^n} > n + \frac{\lambda - b - \beta\lambda}{\lambda - b + \beta\lambda} = M.$$

Since

$$\zeta_0 = q^{-1}(p(z_0)) = \frac{\lambda(p(z_0)-1)}{bp(z_0)-\lambda},$$

we have

$$z_0 p'(z_0) = -m\frac{(1-p(z_0))(\lambda - bp(z_0))}{(\lambda - b)}.$$

Set

$$p(z_0) = \frac{\lambda}{\lambda+b} + \frac{\lambda}{\lambda+b}e^{it},$$

for a fix real t. Using the relations obtained at the above and with a simple computation we deduce that

$$\mathfrak{Re}\left\{p(z_0) + k\frac{z_0 p'(z_0)}{p(z_0)}\right\} = \left(\frac{\lambda(\lambda - b(1+km))}{(\lambda+b)(\lambda-b)}\right)(1+\cos t) + mk\frac{\lambda+b}{2(\lambda-b)}. \qquad (3.1)$$

For completing our proof we consider three cases. If $-\lambda < b \leq 0$ then (3.1) implies that

$$\mathfrak{Re}\left\{p(z_0) + k\frac{z_0 p'(z_0)}{p(z_0)}\right\} > Mk\frac{\lambda+b}{2(\lambda-b)},$$

which contradicts with the assumption. Also for $0 < b < \lambda \leq b(1+kM)$, we put

$$f(x) = mk\frac{\lambda+b}{2(\lambda-b)} + (1+x)\frac{\lambda}{\lambda+b}\frac{\lambda - b(1+km)}{\lambda-b} \qquad (-1 \leq x \leq 1),$$

where $x = \cos t$. It is clear that

$$f'(x) = \frac{\lambda}{\lambda+b}\frac{\lambda - b(1+km)}{\lambda - b} < 0 \qquad (-1 \leq x \leq 1),$$

so

$$f(x) \geq f(1) = \frac{mk(\lambda-b)}{2(\lambda+b)} + \frac{2\lambda}{(\lambda+b)} > \frac{Mk(\lambda-b)}{2(\lambda+b)} + \frac{2\lambda}{(\lambda+b)} \qquad (-1 \leq x \leq 1),$$

which contradicts with the assumption. Ultimately, for the case $0 < b < \frac{\lambda}{1+kM}$ we set

$$g(x) = \frac{\lambda+b}{2} - \frac{\lambda b}{\lambda+b} - \frac{\lambda b}{\lambda+b}x \qquad (-1 \leq x \leq 1),$$

where $x = \cos t$. Now $g'(x) = -\frac{\lambda}{\lambda+b} < 0$, and so for all $-1 \leq x \leq 1$ we have

$$g(x) \geq g(1) = \frac{(\lambda-b)^2}{2(\lambda+b)} > 0.$$

Consequently,

$$\begin{aligned}\mathfrak{Re}\left\{p(z_0) + k\frac{z_0 p'(z_0)}{p(z_0)}\right\} &= \frac{\lambda}{\lambda+b}(1+x) + \left(\frac{mk}{\lambda-b}\right) g(x) \\ &> \frac{Mk}{(\lambda-b)} \frac{(\lambda-b)^2}{2(\lambda+b)} = Mk\frac{\lambda-b}{2(\lambda+b)},\end{aligned}$$

that contradicts with the assumption. Hence the proof is complete. □

Corollary 3.9. *Let $-\lambda < b < \lambda$, $\lambda > 0$ and $k > 0$. Also let $f \in \mathcal{A}_{n,b_1}$ with $\frac{zf'(z)}{f(z)} \neq \frac{2\lambda}{b+\lambda}$ in $\mathbb{U}$ and $0 \leq b_1 \leq \frac{\lambda-b}{n\lambda}$. If for all $z \in \mathbb{U}$*

$$\begin{aligned}&\mathfrak{Re}\left\{(1-k)\frac{zf'(z)}{f(z)} + k\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} \\ &\leq \begin{cases} Mk\frac{\lambda+b}{2(\lambda-b)} & \text{if} \quad -\lambda < b \leq 0, M \geq \frac{2(\lambda-b)}{k(\lambda+b)} \\ \frac{Mk}{2}\frac{\lambda-b}{\lambda+b} + \frac{2\lambda}{\lambda+b} & \text{if} \quad \frac{\lambda}{1+kM} \leq b < \lambda \\ \frac{Mk}{2}\frac{\lambda-b}{\lambda+b} & \text{if} \quad 0 < b < \frac{\lambda}{1+kM}, M \geq \frac{2(\lambda+b)}{k(\lambda-b)}, \end{cases}\end{aligned}$$

then we have

$$\left|\frac{zf'(z)}{f(z)} - \frac{\lambda}{b+\lambda}\right| < \frac{\lambda}{b+\lambda} \qquad z \in \mathbb{U},$$

where $M = n + \frac{\frac{\lambda-b}{\lambda} - nb_1}{\frac{\lambda-b}{\lambda} + nb_1}$.

Proof. Let $p(z) = \frac{zf'(z)}{f(z)}$ then we have $p \in \mathcal{H}_{nb_1}[1, n]$. Therefore by applying Theorem 3.2, and replacing β by nb_1 in this theorem, we obtain the result. □

Remark 3.10. By putting $b_1 = 0$ in the Corollary 3.6, one can observe that this corollary improves and extends the result obtained in [12](see Theorem 3.1 in [12]).

By setting $k = 1$, $b = 1$, $b_1 = \frac{1}{3}$, $\lambda = 3$ and $n = 2$ in Corollary 3.6 we obtain

Corollary 3.11. *Let $f \in \mathcal{A}_{2,\frac{1}{3}}$ with $\frac{zf'(z)}{f(z)} \neq \frac{3}{2}$ in $\mathbb{U}$. If for all z in the open unit disc*

$$\mathfrak{Re}\{1 + \frac{zf''(z)}{f'(z)}\} \leq 2,$$

then we have

$$\left|\frac{zf'(z)}{f(z)} - \frac{3}{4}\right| < \frac{3}{4} \qquad z \in \mathbb{U}.$$

By setting $k = 1$, $b = 1$, $b_1 = \frac{1}{9}$, $\lambda = 3$ and $n = 3$ in Corollary 3.6 we obtain

Corollary 3.12. *Let $f \in \mathcal{A}_{3,\frac{1}{9}}$ with $\frac{zf'(z)}{f(z)} \neq \frac{3}{2}$ in $\mathbb{U}$. If for all z in the open unit disc*

$$\mathfrak{Re}\{1+\frac{zf''(z)}{f'(z)}\} \leq \frac{11}{3},$$

then we have

$$\left|\frac{zf'(z)}{f(z)} - \frac{3}{4}\right| < \frac{3}{4} \qquad z \in \mathbb{U}.$$

By putting $k=1$, $b=3$, $b_1=\frac{2}{5}$, $\lambda=5$ and $n=1$ in Corollary 3.6 we obtain

Corollary 3.13. *Let $f \in \mathcal{A}_{1,\frac{2}{5}}$ with $\frac{zf'(z)}{f(z)} \neq \frac{5}{4}$ in $\mathbb{U}$. If for all z in the open unit disc*

$$\mathfrak{Re}\{1+\frac{zf''(z)}{f'(z)}\} \leq \frac{11}{8},$$

then we have

$$\left|\frac{zf'(z)}{f(z)} - \frac{5}{8}\right| < \frac{5}{8} \qquad z \in \mathbb{U}.$$

By putting $k=1$ and $b=0$ in Corollary 3.6 we obtain

Corollary 3.14. *Let $n \geq 2$ and $0 \leq b_1 \leq \frac{1}{n}$. Also let $f \in \mathcal{A}_{n,b_1}$ with $\frac{zf'(z)}{f(z)} \neq 2$ in $\mathbb{U}$. If for all z in the open unit disc*

$$\mathfrak{Re}\{1+\frac{zf''(z)}{f'(z)}\} \leq \frac{M}{2},$$

then we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \qquad z \in \mathbb{U},$$

where M is defined in the Corollary 3.6.

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Application of Hayman's theorem to directional differential equations with analytic solutions in the unit ball

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Abstract. In this paper, we investigate analytic solutions of higher order linear non-homogeneous directional differential equations whose coefficients are analytic functions in the unit ball. We use methods of theory of analytic functions in the unit ball having bounded L-index in direction, where $L : \mathbb{B}^n \to \mathbb{R}_+$ is a continuous function such that $L(z) > \frac{\beta|\mathbf{b}|}{1-|z|}$ for all $z \in \mathbb{B}^n$, $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ be a fixed direction, $\beta > 1$ is some constant. Our proofs are based on application of inequalities from analog of Hayman's theorem for analytic functions in the unit ball. There are presented growth estimates of their solutions which contains parameters depending on the coefficients of the equations. Also we obtained sufficient conditions that every analytic solution of the equation has bounded L-index in the direction. The deduced results are also new in one-dimensional case, i.e. for functions analytic in the unit disc.

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1. Introduction

B. Lepson [16] introduced a concept of entire function of bounded index as another approach in analytic theory of differential equations. The first paper was closely connected to linear differential equation of infinite order with constant coefficients. But the functions of bounded index have interesting properties [21, 20, 19]: some regular behavior, uniform distribution of zeros, growth estimates, etc. There are many

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approaches to generalize the concept for wider classes of entire functions of single variable because every entire function of bounded index is a function of exponential type. Perhaps, the most successful approach is a concept of function of bounded l-index, proposed by A. Kuzyk and M. Sheremeta [14], where $l : \mathbb{C} \to \mathbb{R}_+$ is a continuous function. For these functions there are known existence theorems [10, 12]: *for an entire function f there exists a function $l : \mathbb{C} \to \mathbb{R}_+$ such that f has bounded l-index if and only if the function f has bounded multiplicities of zeros.*

In the last years, analytic functions of several variables having bounded index are intensively investigated. Main objects of investigations are such function classes: entire functions of several variables [4, 5, 17, 18], functions analytic in a polydisc [2], in a ball [6, 7].

For entire functions and analytic function in a ball there were proposed two approaches to introduce a concept of index boundedness in a multidimensional complex space. They generate so-called functions of bounded L-index in a direction and functions of bounded $\mathbf{L}$-index in joint variables.

In this research, we will consider the first approach, i.e. analytic functions in the unit ball of bounded L-index in direction. A connection between these two approaches is investigated in [8]. We will consider an application of these functions to study properties of analytic solutions of a linear higher order non-homogeneous differential equation with directional derivatives of the following form:

$$g_0(z)\partial_{\mathbf{b}}^p F(z) + g_1(z)\frac{\partial^{p-1}F(z)}{\partial \mathbf{b}^{p-1}} + \cdots + g_p(z)F(z) = h(z). \tag{1.1}$$

Also, we estimated asymptotic behavior of modulus of analytic functions in the unit ball by the function L and the L-index in the direction $\mathbf{b}$.

The linear PDE's (1.1) can easily be turned into ODEs by a suitable change of directional derivative in the direction $\mathbf{b}$ into a canonical direction along a coordinate axis. The cross-terms will vanish and hence an ODE. But the coefficients of the ODE depend on z if we consider a slice $z+t\mathbf{b}$, $z \in \mathbb{B}^n$, $t \in \mathbb{C}$, $|t| < \frac{1-|z|}{|\mathbf{b}|}$. Therefore, all one-dimensional results need uniform estimates in z. Let us consider an entire function $F(z_1, z_2) = \cos\sqrt{z_1z_2} = \sum_{n=0}^{\infty}\frac{(-1)^n(z_1z_2)^{2n}}{n!}$. It is known [4] that for fixed $z^0 = (z_1^0, z_2^0)$ and for every $\mathbf{b} = (b_1, b_2) \in \mathbb{C}^2 \setminus \{0\}$ the function $F(z_1^0 + tb_1, z^0 + tb_2)$ has bounded index as a function of variable $t \in \mathbb{C}$. But the function $F(z_1, z_2)$ has unbounded index in any direction $\mathbf{b}$ because the indexes of the function $F(z_1^0 + tb_1, z^0 + tb_2)$ are not uniformly bounded in (z_1^0, z_2^0). This fact was proved with the application of differential equations in [4]. It shows that uniform estimates play an important role and the bounded index in direction can not be simply reduced to the one-dimensional bounded index.

Our results are generalizations of earlier obtained results for entire functions of bounded L-index in a direction [3]. But now we consider the function L of more general form than in [3]. There was considered only $L(z) = l(|z|)$, where $l : \mathbb{R}_+ \to \mathbb{R}_+$, and $z \in \mathbb{C}^n$. In addition to Hayman's theorem, L-index boundedness in direction of analytic solutions of partial differential equations can be established by the so-called logarithmic criterion (see [9])). This approach requires that all coefficients of the equations have bounded L-index in the direction. For entire functions of one variable having bounded l-index the similar results were deduced in [15, 11].

Let us introduce some notations and definitions.

Note that the positivity and continuity of the function L are weak restrictions to deduce constructive results. Thus, we assume additional restrictions by the function L.

Let $\mathbf{0} = (0,\dots,0)$, $\mathbf{b} = (b_1,\dots,b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direction, $\mathbb{R}_+ = (0,+\infty)$, $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$, $L : \mathbb{B}^n \to \mathbb{R}_+$ be a continuous function such that for all $z \in \mathbb{B}^n$

$$L(z) > \frac{\beta|\mathbf{b}|}{1-|z|},\ \beta = \text{const} > 1. \tag{1.2}$$

Analytic function $F : \mathbb{B}^n \to \mathbb{C}$ is called [7] a function of *bounded L-index in a direction* $\mathbf{b}$ if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and every $z \in \mathbb{B}^n$ the following inequality is valid

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m!L^m(z)} \le \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)} : 0 \le k \le m_0\right\}, \tag{1.3}$$

where $\partial_{\mathbf{b}}^0 F(z) = F(z), \partial_{\mathbf{b}} F(z) = \sum\limits_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j$, $\partial_{\mathbf{b}}^k F(z) = \partial_{\mathbf{b}}\left(\partial_{\mathbf{b}}^{k-1} F(z)\right)$, $k \ge 2$.

The least such integer $m_0 = m_0(\mathbf{b})$ is called the *L-index in the direction* $\mathbf{b}$ *of the analytic function F and is denoted by* $N_{\mathbf{b}}(F,L) = m_0$. If $n = 1$, $\mathbf{b} = 1$, $L = l$, $F = f$, then $N(f,l) \equiv N_1(f,l)$ is called the l-index of the function f. In the case $n = 1$ and $\mathbf{b} = 1$ we obtain the definition of an analytic function in the unit disc of bounded l-index. The definition is a generalization of concept of bounded L-index in direction introduced and considered for entire functions of several variables in [5]. The primary definition of bounded index for entire function of one variable was defined by B. Lepson [16].

Let $\mathbb{D} = \{t \in \mathbb{C} : |t| < 1\}$, $L : \mathbb{B}^n \to \mathbb{R}_+$ be a continuous function. For $z \in \mathbb{B}^n$ we denote $D_z = \{t \in \mathbb{C} : |t| \le \frac{1-|z|}{|\mathbf{b}|}\}$,

$$\lambda_{\mathbf{b}}(\eta) = \sup_{z\in\mathbb{B}^n} \sup_{t_1,t_2\in D_z} \left\{\frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})} : |t_1-t_2| \le \frac{\eta}{\min\{L(z+t_1\mathbf{b}), L(z+t_2\mathbf{b})\}}\right\}.$$

The notation $Q_{\mathbf{b}}(\mathbb{B}^n)$ stands for a class of positive continuous functions $L : \mathbb{B}^n \to \mathbb{R}_+$, satisfying (1.2) and

$$(\forall \eta \in [0,\beta]) :\ \lambda_{\mathbf{b}}(\eta) < +\infty. \tag{1.4}$$

If $n = 1$ then $Q(\mathbb{D}) \equiv Q_1(\mathbb{B}^1)$ and $\lambda(\eta) \equiv \lambda_1(\eta)$.

Let D be an arbitrary bounded domain in $\mathbb{B}^n$ such that $\operatorname{dist}(D,\mathbb{B}^n) > 0$. If inequality (1.3) holds for all $z \in D$ instead $\mathbb{B}^n$, then the analytic function $F : \mathbb{B}^n \to \mathbb{C}$ is called *a function of bounded L-index in the direction* $\mathbf{b}$ *in the domain* D. The least such integer m_0 is called the *L-index in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *in domain* D and is denoted by $N_{\mathbf{b}}(F,L,D) = m_0$. The notation $\overline{D}$ stands for a closure of the domain D.

Lemma 1.1 ([1])**.** *Let D be an arbitrary bounded domain in $\mathbb{B}^n$ such that*

$$d = \operatorname{dist}(D,\mathbb{B}^n) = \inf_{z\in D}(1-|z|) > 0,\ \beta > 1,\ \mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$$

be an arbitrary direction. If $L\colon \mathbb{B}^n \to \mathbb{R}_+$ *is continuous function such that* $L(z) \geq \frac{\beta|\mathbf{b}|}{d}$, *and* $F : \mathbb{B}^n \to \mathbb{C}$ *is analytic function such that* $(\forall z^0 \in \overline{D})$: $\quad F(z^0 + t\mathbf{b}) \not\equiv 0$, *then* $N_{\mathbf{b}}(F, L, D) < \infty$.

Below we present an analog of Hayman's Theorem [13]. The theorem helps to investigate boundedness L-index in direction of analytic solutions of differential equations. At the end of the paper, we will present a scheme of this application.

Theorem 1.2 ([7]). *Let* $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. *An analytic function* $F : \mathbb{B}^n \to \mathbb{C}$ *is of bounded* L*-index in the direction* $\mathbf{b}$ *if and only if there exist* $p \in \mathbb{Z}_+$ *and* $C > 0$ *such that for every* $z \in \mathbb{B}^n$

$$\left|\frac{\partial_{\mathbf{b}}^{p+1} F(z)}{L^{p+1}(z)}\right| \leq C \max\left\{\left|\frac{\partial_{\mathbf{b}}^{k} F(z)}{L^{k}(z)}\right| :\ 0 \leq k \leq p\right\}. \tag{1.5}$$

Using Lemma 1.1, we yield the following corollary with this criterion.

Corollary 1.3. *Let* $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, G *be a domain compactly embedded in* $\mathbb{B}^n$ *such that* $d = \operatorname{dist}(D, \mathbb{B}^n) = \inf_{z \in D}(1 - |z|) > 0$ *and for all* $z \in G \quad L(z) \geq \frac{\beta|\mathbf{b}|}{d}$. *An analytic function* $F : \mathbb{B}^n \to \mathbb{C}$ *has bounded* L*-index in the direction* $\mathbf{b}$ *if and only if there exist* $p \in \mathbb{Z}_+$ *and* $C > 0$ *such that for all* $z \in \mathbb{B}^n \setminus G$ *the following relation holds*

$$\frac{|\partial_{\mathbf{b}}^{p+1} F(z)|}{L^{p+1}(z)} \leq C \max\left\{\frac{|\partial_{\mathbf{b}}^{k} F(z)|}{L^{k}(z)} :\ 0 \leq k \leq p\right\}. \tag{1.6}$$

2. Auxiliary lemmas

We denote $a^+ = \max\{a, 0\}$. Set $u(r) = u(z^0, \theta, r) = L(z^0 + re^{i\theta}\mathbf{b})$. Let $W_{\mathbf{b}}(\mathbb{B}^n)$ be a class of positive continuous function $L : \mathbb{B}^n \to \mathbb{R}_+$ satisfying all following conditions:

1) for all $z \in \mathbb{B}^n \quad L(z) > \frac{\beta|\mathbf{b}|}{1-|z|}$, where $\beta = \text{const} > 1$;
2) for every $z^0 \in \mathbb{B}^n$ and every $\theta \in [0, 2\pi]$ the function $u(r, z^0, \theta)$ be a continuously differentiable function of real variable $r \in [0, r_0)$, where
$$r_0 = \min\{s \in \mathbb{R}_+ :\ |z^0 + se^{i\theta}\mathbf{b}| = 1\};$$
3) for every $z^0 \in \mathbb{B}^n$, $\theta \in [0, 2\pi]$ one has
$$\left(\frac{d}{ds}\frac{1}{L(z^0 + sre^{i\theta}\mathbf{b})}\Big|_{s=1}\right)^+ \to 0 \text{ as } |z^0 + re^{i\theta}\mathbf{b}| \to 1, \text{ i.e. } r \to r_0.$$

The conditions 2) and 3) together can be replaced by some strict condition $\partial_{\mathbf{b}}(1/L(Re^{i\Theta})) \to 0$ as $|R| \to 1$, where $Re^{i\Theta} = (r_1e^{i\theta_1}, \ldots, r_ne^{i\theta_n})$, $|R| < 1$, $\theta_j \in [0, 2\pi]$, and $L(Re^{i\Theta})$ is positive continuously differentiable function in each variable r_j, $j \in \{1, \ldots, n\}$. Moreover, condition 3) is equivalent to $\frac{(-u_r'(z^0,\theta,r))^+}{L^2(z^0+re^{i\theta}\mathbf{b})} \to 0$ as $r \to r_0$. Beside, condition 1) yields that $\int\limits_0^r L(z^0 + xe^{i\theta}\mathbf{b})dx \to +\infty$ as $|z^0 + re^{i\theta}\mathbf{b}| \to 1$.

First, we prove the following two lemmas. For entire functions of bounded L-index in direction they were obtained in [3].

Lemma 2.1. *Let $L \in W_{\mathbf{b}}(\mathbb{B}^n)$, $F : \mathbb{B}^n \to \mathbb{C}$ be an analytic function such that $\exists R \in [0,1)$ $\forall z \in \mathbb{B}^n$ $|z| < R$ one has $F(z+t\mathbf{b}) \not\equiv 0$. If there exist numbers $p \in \mathbb{Z}_+$, $C > 0$ such that for all $z \in \mathbb{B}^n$, $|z| \geq R$, the inequality*

$$\frac{|\partial_{\mathbf{b}}^{p+1} F(z)|}{L^{p+1}(z)} \leq C \max\left\{ \frac{|\partial_{\mathbf{b}}^{k} F(z)|}{L^{k}(z)} : 0 \leq k \leq p \right\}, \tag{2.1}$$

holds then for every $z^0 \in \mathbb{B}^n$ and for every $\theta \in [0, 2\pi]$

$$\varlimsup_{|z^0+re^{i\theta}\mathbf{b}|\to 1} \frac{\ln |F(z^0+re^{i\theta}\mathbf{b})|}{\int_0^r L(z^0+xe^{i\theta}\mathbf{b})dx} \leq \max\{1, C\}.$$

Proof. Let $\theta \in [0, 2\pi]$, $z^0 \in \mathbb{B}^n$ be fixed and $x \in [0, r_0)$ be such that $|z^0 + xe^{i\theta}\mathbf{b}| \geq R$. We define

$$\Omega_{z^0}(x) = \max\left\{ \frac{|\partial_{\mathbf{b}}^{k} F(z^0+xe^{i\theta}\mathbf{b})|}{L^{k}(z^0+xe^{i\theta}\mathbf{b})} : 0 \leq k \leq p \right\}.$$

The function $\frac{|\partial_{\mathbf{b}}^{k} F(z^0+xe^{i\theta}\mathbf{b})|}{L^{k}(z^0+xe^{i\theta}\mathbf{b})}$ is continuously differentiable by real $x \in [0, r^*]$, outside the zero set of function $|\partial_{\mathbf{b}}^{k} F(z^0+xe^{i\theta}\mathbf{b})|$ because $L \in W_{\mathbf{b}}(\mathbb{B}^n)$. Thus, the function $\Omega_{z^0}(x)$ is a continuously differentiable function on $[0, r^*]$, apart from, possibly, a countable set. For absolutely continuous functions h_1, h_2, $\ldots$, h_k and $h(x) := \max\{h_j(z) : 1 \leq j \leq k\}$, $h'(x) \leq \max\{h_j'(x) : 1 \leq j \leq k\}$, $x \in [a, b]$ (see [21, Lemma 4.1, p. 81]). The function $\Omega_{z^0}(x)$ is absolutely continuous. Therefore,

$$\Omega_{z^0}'(x) \leq \max\left\{ \frac{d}{dx}\left(\frac{1}{L^{k}(z^0+xe^{i\theta}\mathbf{b})} \cdot \left|\partial_{\mathbf{b}}^{k} F(z^0+xe^{i\theta}\mathbf{b})\right| \right) : 0 \leq k \leq p \right\}$$

except on a countable set of points.

Using the inequality $\frac{d}{dx}|\varphi(x)| \leq \left|\frac{d}{dx}\varphi(x)\right|$, which holds for complex-valued function of real argument except at the points $x = t$ such that $\varphi(t) = 0$, in view of (2.1) we obtain

$$\begin{aligned}
\Omega_{z^0}'(x) \leq \max_{0\leq k\leq p} \Big\{ |e^{i\theta}| \frac{1}{L^{k}(z^0+xe^{i\theta}\mathbf{b})} \left|\partial_{\mathbf{b}}^{k+1} F(z^0+xe^{i\theta}\mathbf{b})\right| - \left|\partial_{\mathbf{b}}^{k} F(z^0+xe^{i\theta}\mathbf{b})\right| \times \\
\times \frac{k \cdot u_x'(z^0, \theta, x)}{L^{k+1}(z^0+xe^{i\theta}\mathbf{b})} \Big\} \leq \\
\leq \max_{0\leq k\leq p} \Big\{ \frac{1}{L^{k+1}(z^0+xe^{i\theta}\mathbf{b})} \left|\partial_{\mathbf{b}}^{k+1} F(z^0+xe^{i\theta}\mathbf{b})\right| L(z^0+xe^{i\theta}\mathbf{b}) - \\
- \frac{1}{L^{k}(z^0+xe^{i\theta}\mathbf{b})} \left|\partial_{\mathbf{b}}^{k} F(z^0+xe^{i\theta}\mathbf{b})\right| \frac{k u_x'(z^0, \theta, x)}{L(z^0+xe^{i\theta}\mathbf{b})} \Big\} \leq \\
\leq \Omega_{z^0}(x) \left(CL(z^0+xe^{i\theta}\mathbf{b}) + p\frac{(-u_x'(z^0,\theta,x))^+}{L(z^0+xe^{i\theta}\mathbf{b})} \right).
\end{aligned}$$

From condition 3) in the definition of the class $W_{\mathbf{b}}(\mathbb{B}^n)$ we have

$$u_x'(z^0, \theta, x) = o(L^2(z^0+xe^{i\theta}\mathbf{b})) \text{ as } x \to r_0,$$

then

$$\Omega'_{z^0}(x) \le \Omega_{z^0}(x)(\max\{1,C\}L(z^0+xe^{i\theta}\mathbf{b})+p\varepsilon L(z^0+xe^{i\theta}\mathbf{b})) \le \Omega_{z^0}(x)\times$$
$$\times L(z^0+xe^{i\theta}\mathbf{b})(\max\{1,C\}+p\varepsilon) \le \Omega_{z^0}(x)L(z^0+xe^{i\theta}\mathbf{b})\max\{1,C\}(1+p\varepsilon)$$

for all $\varepsilon>0$ and for all $x\in[x_0(z^0,\theta,\varepsilon),r_0)$ outside a countable set of points for given $z^0\in\mathbb{B}^n$ and $\theta\in[0,2\pi]$ Hence, there exists $r_1\ge x_0(z^0,\theta,\varepsilon)$ such that

$$\Omega_{z^0}(r) \le \Omega_{z^0}(r_1)\cdot\exp\left\{(1+p\varepsilon)\max\{1,C\}\int_{r_1}^{r} L(z^0+xe^{i\theta}\mathbf{b})dx\right\}$$

for every $r\in[r_1,r_0)$. From the definition of $\Omega_{z^0}(x)$ for $k=0$ we obtain that

$$|F(z^0+re^{i\theta}\mathbf{b})| \le \Omega_{z^0}(r_1)\cdot\exp\left\{(1+p\varepsilon)\max\{1,C\}\int_{0}^{r} L(z^0+xe^{i\theta}\mathbf{b})dx\right\},$$

$$\ln|F(z^0+re^{i\theta}\mathbf{b})| \le \ln\Omega_{z^0}(r_1)+(1+p\varepsilon)\max\{1,C\}\int_{0}^{r} L(z^0+xe^{i\theta}\mathbf{b})dx,$$

$$\frac{\ln|F(z^0+re^{i\theta}\mathbf{b})|}{\int_0^r L(z^0+xe^{i\theta}\mathbf{b})dx} \le \frac{\ln\Omega_{z^0}(r_1)}{\int_0^r L(z^0+xe^{i\theta}\mathbf{b})dx}+(1+p\varepsilon)\max\{1,C\}.$$

From this inequality for all $z^0\in\mathbb{B}^n$ and $\theta\in[0,2\pi]$ we obtain that

$$\varlimsup_{|z^0+re^{i\theta}\mathbf{b}|\to1}\frac{\ln|F(z^0+re^{i\theta}\mathbf{b})|}{\int\limits_0^r L(z^0+xe^{i\theta}\mathbf{b})dx} \le \max\{1,C\}. \qquad \square$$

Lemma 2.2. *Let $L\in W_{\mathbf{b}}(\mathbb{B}^n)$, $F:\mathbb{B}^n\to\mathbb{C}$ be an analytic function such that $\exists R\in[0,1)$ $\forall z\in\mathbb{B}^n$ $|z|<R$ one has $F(z+t\mathbf{b})\not\equiv0$. If there exist numbers $p\in\mathbb{Z}_+$, $C>0$ such that for all $z\in\mathbb{B}^n$, $|z|\ge R$, the inequality*

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{(p+1)!L^{p+1}(z)} \le C\max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{k!L^{k}(z)}:0\le k\le p\right\}, \tag{2.2}$$

holds then for all $z^0\in\mathbb{B}^n$ $\varlimsup\limits_{|z^0+re^{i\theta}\mathbf{b}|\to1}\frac{\ln|F(z^0+re^{i\theta}\mathbf{b})|}{\int\limits_0^r L(z^0+xe^{i\theta}\mathbf{b})dx}\le(p+1)\max\{1,C\}.$

Proof. Let $\theta\in[0,2\pi]$, $z^0\in\mathbb{B}^n$ be fixed and $x\in\mathbb{R}_+$ be such that $|z^0+xe^{i\theta}\mathbf{b}|\ge R$. We denote

$$\Omega_{z^0}(x)=\max\left\{\frac{1}{k!L^k(z^0+xe^{i\theta}\mathbf{b})}\cdot\left|\partial_{\mathbf{b}}^{k}F(z^0+xe^{i\theta}\mathbf{b})\right|:0\le k\le p\right\}.$$

As in Lemma 2.1 the function $\Omega_{z^0}(x)$ is continuously differentiable because $L\in W_{\mathbf{b}}(\mathbb{B}^n)$ and

$$\Omega'_{z^0}(x) \le \max\left\{\frac{d}{dx}\left(\frac{1}{k!L^k(z^0+xe^{i\theta}\mathbf{b})}\left|\partial_{\mathbf{b}}^{k}F(z^0+xe^{i\theta}\mathbf{b})\right|\right):0\le k\le p\right\}$$

except a countable set of points. Applying the inequality $\frac{d}{dx}|\varphi(x)|\le\left|\frac{d}{dx}\varphi(x)\right|$, which holds for complex-valued function of real argument outside a countable set of points, in view of (2.2) we obtain

$$\Omega'_{z^0}(x) \le \max\Big\{|e^{i\theta}|\frac{1}{k!L^k(z^0+xe^{i\theta}\mathbf{b})}\left|\partial_{\mathbf{b}}^{k+1}F(z^0+xe^{i\theta}\mathbf{b})\right| -$$
$$-\left|\partial_{\mathbf{b}}^{k}F(z^0+xe^{i\theta}\mathbf{b})\right|\frac{ku'_x(z^0,\theta,x)}{k!L^{k+1}(z^0+xe^{i\theta}\mathbf{b})} : 0\le k\le p\Big\} \le$$
$$\le \max\left\{\frac{1}{(k+1)!L^{k+1}(z^0+xe^{i\theta}\mathbf{b})}\left|\partial_{\mathbf{b}}^{k+1}F(z^0+xe^{i\theta}\mathbf{b})\right|(k+1)L(z^0+xe^{i\theta}\mathbf{b})-\right.$$
$$\left.-\frac{1}{L^k(z^0+xe^{i\theta}\mathbf{b})k!}\left|\partial_{\mathbf{b}}^{k}F(z^0+xe^{i\theta}\mathbf{b})\right|\frac{ku'_x(z^0,\theta,x)}{L(z^0+xe^{i\theta}\mathbf{b})} : 0\le k\le p\right\} \le$$
$$\le \Omega_{z^0}(x)\left(\max\{1,C\}L(z^0+xe^{i\theta}\mathbf{b})(p+1)+p\frac{(-u'_x(z^0,\theta,x))^+}{L(z^0+xe^{i\theta}\mathbf{b})}\right).$$

But we have that $L\in W_{\mathbf{b}}(\mathbb{B}^n)$, i.e. $\frac{(-u'_x(z^0,\theta,x))^+}{L^2(z^0+xe^{i\theta}\mathbf{b})} \to 0$ as $|z^0+xe^{i\theta}\mathbf{b}|\to 1$. Therefore,

$$\Omega'_{z^0}(x) \le \Omega_{z^0}(x)(\max\{1,C\}L(z^0+xe^{i\theta}\mathbf{b})(p+1)+p\varepsilon L(z^0+xe^{i\theta}\mathbf{b})) \le$$
$$\le \Omega_{z^0}(x)L(z^0+xe^{i\theta}\mathbf{b})(\max\{1,C\}(p+1)+p\varepsilon) \le \Omega_{z^0}(x)L(z^0+xe^{i\theta}\mathbf{b})\times$$
$$\times \max\{1,C\}(p+1)\left(1+\frac{p}{p+1}\varepsilon\right)$$

for all $\varepsilon>0$ and for all $x\ge x_0(z^0,\theta,\varepsilon)$, except a countable set of points at given z^0 and θ. Thus, there exists $r_1\ge x_0(z^0,\theta,\varepsilon)$ that for $r>r_1$ we have

$$\Omega_{z^0}(r)\le \Omega_{z^0}(r_1)\cdot\exp\left\{(1+\varepsilon)\max\{1,C\}(p+1)\int_{r_1}^{r}L(z^0+xe^{i\theta}\mathbf{b})dx\right\}.$$

Be definition of $\Omega_{z^0}(x)$ at $k=0$ we obtain

$$|F(z^0+re^{i\theta}\mathbf{b})|\le \Omega_{z^0}(r_0)\exp\left\{(1+\varepsilon)\max\{1,C\}(p+1)\int_{0}^{r}L(z^0+xe^{i\theta}\mathbf{b})dx\right\}.$$

Therefore,

$$\ln|F(z^0+re^{i\theta}\mathbf{b})|\le \ln\Omega_{z^0}(r_0)+(1+\varepsilon)\max\{1,C\}\int_{0}^{r}L(z^0+xe^{i\theta}\mathbf{b})dx.$$

Dividing of the inequality by $\int\limits_0^r L(z^0+xe^{i\theta}\mathbf{b})dx$, we obtain

$$\frac{\ln|F(z^0+e^{i\theta}r\mathbf{b})|}{\int\limits_0^r L(z^0+xe^{i\theta}\mathbf{b})dx}\le \frac{\ln\Omega_{z^0}(r_0)}{\int\limits_0^r L(z^0+xe^{i\theta}\mathbf{b})dx}+(1+\varepsilon)\max\{1,C\}(p+1).$$

Thus, for all $z\in\mathbb{B}^n$ and $\theta\in[0,2\pi]$ we obtain an estimate

$$\overline{\lim_{|z^0+re^{i\theta}\mathbf{b}|\to 1}}\frac{\ln|F(z^0+e^{i\theta}r\mathbf{b})|}{\int\limits_0^r L(z^0+xe^{i\theta}\mathbf{b})dx}\le \max\{1,C\}(p+1). \qquad \square$$

Remark 2.3. Note that condition (2.2) means that

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L^{p+1}(z)} =$$

$$= (p+1)! \cdot \frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{(p+1)!L^{p+1}(z)} \leq (p+1)!C\max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{k!L^{k}(z)} : 0 \leq k \leq p\right\} \leq$$

$$\leq (p+1)!C \cdot \max\left\{\frac{1}{k!} : 0 \leq k \leq p\right\} \cdot \max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{L^{k}(z)} : 0 \leq k \leq p\right\} \leq$$

$$\leq (p+1)!C\max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{L^{k}(z)} : 0 \leq k \leq p\right\}.$$

Hence, by Lemma 2.1 we have

$$\varlimsup_{|z^0+re^{i\theta}\mathbf{b}|\to 1} \frac{\ln|F(z^0+e^{i\theta}r\mathbf{b})|}{\int\limits_0^r L(z^0+xe^{i\theta}\mathbf{b})dx} \leq \max\{1, C(p+1)!\}.$$

Since $c(p+1)! > c(p+1)$ for $p > 1$, we see that Lemma 2.2 does not imply growth estimate of Lemma 2.1. Clearly, Lemma 2.1 does not imply Lemma 2.2 as well. Therefore, we need both Lemma 2.1 and Lemma 2.2.

3. Growth and bounded L-index in direction of analytic solutions of partial differential equations

Using proved lemmas we formulate and prove propositions that provide growth estimates of analytic solutions of the partial differential equation

$$g_0(z)\partial_{\mathbf{b}}^{p}F(z) + g_1(z)\frac{\partial^{p-1}F(z)}{\partial\mathbf{b}^{p-1}} + \cdots + g_p(z)F(z) = h(z). \tag{3.1}$$

Let us denote $QW_{\mathbf{b}}(\mathbb{B}^n) = Q_{\mathbf{b}}(\mathbb{B}^n) \cap W_{\mathbf{b}}(\mathbb{B}^n)$.

Theorem 3.1. *Let $L \in QW_{\mathbf{b}}(\mathbb{B}^n)$, functions g_0, g_1,.., g_p, and h be analytic in the unit ball and there exists $R \in [0,1)$ such that for all $z \in \mathbb{B}^n$, $|z| \geq R$, the following conditions hold*

1) $|g_j(z)| \leq m_j L^j(z)|g_0(z)|$ *for* $1 \leq j \leq p$;
2) $|\partial_{\mathbf{b}} g_j(z)| < M_j \cdot L^{j+1}(z)|g_0(z)|$ *for* $0 \leq j \leq p$;
3) $|\partial_{\mathbf{b}} h(z)| \leq M \cdot L(z) \cdot |h(z)|$,

where m_j and M are nonnegative constants and M_j are positive constants. If an analytic function $F : \mathbb{B}^n \to \mathbb{C}$ satisfies equation (3.1) *and $\forall z \in \mathbb{B}^n$, $|z| < R$,*

$$F(z+t\mathbf{b}) \not\equiv 0$$

then F has bounded L-index in the direction $\mathbf{b}$ and for all $z^0 \in \mathbb{B}^n$, $\theta \in [0, 2\pi]$

$$\varlimsup_{|z^0+re^{i\theta}\mathbf{b}|\to 1} \frac{\ln|F(z^0+e^{i\theta}r\mathbf{b})|}{\int\limits_0^r L(z^0+te^{i\theta}\mathbf{b})dt} \leq \max\{1, C\}, \tag{3.2}$$

where

$$C=\sum_{j=1}^{p} M_j+(M+1)\sum_{j=1}^{p} m_j+M.$$

Proof. First, we note that the second condition of the theorem with $j=0$ implies that

$$g_0(z)\neq 0 \text{ for } z\in\mathbb{B}^n,\ |z|\geq R.$$

Taking into account that the function $F(z)$ satisfies equation (3.1), we calculate the derivative in the direction $\mathbf{b}$ for this equation

$$g_0(z)\partial_{\mathbf{b}}^{p+1}F(z)+\sum_{j=0}^{p}\partial_{\mathbf{b}}g_j(z)\cdot\partial_{\mathbf{b}}^{p-j}F(z)+\sum_{j=1}^{p}g_j(z)\partial_{\mathbf{b}}^{p-j+1}F(z)+$$
$$+\sum_{j=1}^{p}g_j(z)\partial_{\mathbf{b}}^{p-j+1}F(z)=\partial_{\mathbf{b}}h(z). \tag{3.3}$$

Using the third condition of the theorem, we obtain

$$|\partial_{\mathbf{b}}h(z)|\leq ML(z)h(z)\leq ML(z)\sum_{j=0}^{p}|g_j(z)|\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|. \tag{3.4}$$

By (3.3)

$$\partial_{\mathbf{b}}^{p+1}F(z)=\frac{1}{g_0(z)}\left(\partial_{\mathbf{b}}h(z)-\sum_{j=0}^{p}\partial_{\mathbf{b}}g_j(z)\cdot\partial_{\mathbf{b}}^{p-j}F(z)-\sum_{j=1}^{p}g_j(z)\partial_{\mathbf{b}}^{p-j+1}F(z)\right). \tag{3.5}$$

Putting in the first condition of the theorem $m_0=1$, from (3.5) in view of the second condition we obtain

$$\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|\leq\frac{1}{g_0(z)}\left(ML(z)\sum_{j=0}^{p}|g_j(z)|\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\sum_{j=0}^{p}|\partial_{\mathbf{b}}g_j(z)|\times\right.$$
$$\left.\times\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\sum_{j=1}^{p}|g_j(z)|\left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right|\right)\leq$$
$$\leq ML(z)\sum_{j=0}^{p}m_jL^j(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+$$
$$+\sum_{j=0}^{p}M_jL^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\sum_{j=1}^{p}m_jL^j(z)\left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right|.$$

Dividing this inequality by $L^{p+1}(z)$, we obtain

$$\frac{1}{L^{p+1}(z)}\left|\partial_{\mathbf{b}}^{p+1}F(z)\right| \le M\sum_{j=0}^{p} m_j \frac{1}{L^{p-j}(z)}\left|\partial_{\mathbf{b}}^{p-j}F(z)\right| + \sum_{j=0}^{p} M_j \frac{1}{L^{p-j}(z)}\times$$
$$\times\left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right| + \sum_{j=1}^{p} m_j \frac{1}{L^{p-j+1}(z)}\left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right| \le \Bigg(M\sum_{j=0}^{p} m_j +$$
$$+\sum_{j=0}^{p} M_j + \sum_{j=1}^{p} m_j\Bigg)\max\left\{\frac{1}{L^k(z)}\left|\partial_{\mathbf{b}}^{k}F(z)\right| : 0\le k\le p\right\} =$$
$$= \left((M+1)\sum_{j=1}^{p} m_j + \sum_{j=0}^{p} M_j + M\right)\max\left\{\frac{1}{L^k(z)}\left|\partial_{\mathbf{b}}^{k}F(z)\right| : 0\le k\le p\right\}$$

for all $z\in\mathbb{B}^n$, $|z|\ge R$.

Thus, by Lemma 2.1 estimate (3.2) holds, and by Corollary 1.3 the analytic function $F(z)$ is of bounded L-index in the direction b. □

In the case when equation (3.1) is homogeneous ($h(z)\equiv 0$), the previous theorem can be simplified.

Theorem 3.2. *Let $L\in QW_{\mathbf{b}}(\mathbb{B}^n)$, functions g_0, $g_1, .., g_p$ be analytic in the unit ball and there exists $R\in[0,1)$ such that for all $z\in\mathbb{B}^n$, $|z|\ge R$, one has $|g_j(z)|\le m_jL^j(z)|g_0(z)|$ for $1\le j\le p$, where m_j are some nonnegative constants. If an analytic function $F:\mathbb{B}^n\to\mathbb{C}$ satisfies equation (3.1) with $h(z)\equiv 0$ and $\forall z\in\mathbb{B}^n$, $|z|<R$, $F(z+t\mathbf{b})\not\equiv 0$ then $F(z)$ is of bounded L-index in the direction $\mathbf{b}$ and for all $z^0\in\mathbb{B}^n$, $\theta\in[0,2\pi]$*

$$\overline{\lim_{|z^0+re^{i\theta}\mathbf{b}|\to 1}} \frac{\ln|F(z^0+e^{i\theta}r\mathbf{b})|}{\int_0^r L(z^0+te^{i\theta}\mathbf{b})dt} \le \max\left\{1, \sum_{j=1}^{p} m_j\right\}. \qquad (3.6)$$

Proof. Equation (3.1) implies $g_0(z)\partial_{\mathbf{b}}^{p}F(z) = -\sum_{j=1}^{p} g_j(z)\partial_{\mathbf{b}}^{p-j}F(z)$. Then

$$|g_0(z)|\,|\partial_{\mathbf{b}}^{p}F(z)| \le \sum_{j=1}^{p}|g_j(z)|\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|.$$

Dividing the obtained inequality by $g_0(z)L^p(z)$ and using assumptions of the theorem by the functions $g_j(z)$, we obtain

$$\frac{1}{L^p(z)}|\partial_{\mathbf{b}}^{p}F(z)| \le \sum_{j=1}^{p}\left|\frac{g_j(z)}{g_0(z)}\right|\frac{1}{L^p(z)}\left|\partial_{\mathbf{b}}^{p-j}F(z)\right| \le \sum_{j=1}^{p}\frac{m_j}{L^{p-j}(z)}\times$$
$$\times\left|\partial_{\mathbf{b}}^{p-j}F(z)\right| \le \sum_{j=1}^{p} m_j\max\left\{\frac{1}{L^k(z)}\left|\partial_{\mathbf{b}}^{k}F(z)\right| : 0\le k\le p-1\right\}.$$

Thus, all conditions of Corollary 1.3 are obeyed. Hence, the function F is of bounded L-index in the direction $\mathbf{b}$ and by Lemma 2.1 estimate (3.6) is true. □

Moreover, using Corollary 1.3 and Lemma 2.2 we can complement two previous Theorems 3.1 and 3.2 by propositions, that contain growth estimates, which can sometimes be better than (3.6) and (3.2). Two following theorems have similar proofs as in Theorems 3.1 and 3.2.

Theorem 3.3. *Let $L \in QW_{\mathbf{b}}(\mathbb{B}^n)$, functions g_0, $g_1, .., g_p$, and h be analytic in the unit ball and there exists $R \in [0,1)$ such that for all $z \in \mathbb{B}^n$, $|z| \geq R$, the following conditions hold*

1. $|g_j(z)| \leq m_j L^j(z)|g_0(z)|$ *for* $1 \leq j \leq p$;
2. $|\partial_{\mathbf{b}} g_j(z)| < M_j \cdot L^{j+1}(z)|g_0(z)|$ *for* $0 \leq j \leq p$;
3. $|\partial_{\mathbf{b}} h(z)| \leq M \cdot L(z) \cdot |h(z)|$,

where m_j and M are nonnegative constants and M_j are positive constants. If an analytic function $F : \mathbb{B}^n \to \mathbb{C}$ is a solution of equation (3.1) and $\forall z \in \mathbb{B}^n$, $|z| < R$, $F(z + t\mathbf{b}) \not\equiv 0$ then $F(z)$ is of bounded L-index in the direction $\mathbf{b}$ and for all $z^0 \in \mathbb{B}^n$, $\theta \in [0, 2\pi]$

$$\overline{\lim_{|z^0+re^{i\theta}\mathbf{b}|\to 1}} \frac{\ln |F(z^0 + e^{i\theta} r\mathbf{b})|}{\int\limits_0^r L(z^0 + te^{i\theta}\mathbf{b})dt} \leq \max\{p + 1, 2(M + 2)M^*\}, \tag{3.7}$$

where $M^* = \max\{1, m_j, M_j\}$.

Proof. First, we note that the second condition of this theorem when $j = 0$ implies that $g_0(z) \neq 0$ for $z \in \mathbb{B}^n$, $|z| \geq R$, because in this case we have

$$\left|\frac{\partial g_0(z)}{\partial \mathbf{b}}\right| < M_0 L(z) g_0(z).$$

Since the function $F(z)$ satisfies the equation (3.1), then we calculate a derivative of this equation in the direction $\mathbf{b}$:

$$g_0(z)\partial_{\mathbf{b}}^{p+1} F(z) + \sum_{j=0}^{p} \frac{\partial_{\mathbf{b}} g_j(z)}{\cdot} \partial_{\mathbf{b}}^{p-j} F(z) + \sum_{j=1}^{p} g_j(z)\partial_{\mathbf{b}}^{p-j+1} F(z) +$$
$$+ \sum_{j=1}^{p} g_j(z)\partial_{\mathbf{b}}^{p-j+1} F(z) = \frac{\partial h(z)}{\partial \mathbf{b}}. \tag{3.8}$$

Using the third condition of this theorem, we obtain

$$|\partial_{\mathbf{b}} h(z)| \leq ML(z)|h(z)| \leq ML(z) \sum_{j=0}^{p} |g_j(z)| \left|\partial_{\mathbf{b}}^{p-j} F(z)\right|.$$

From (3.8) it follows

$$\partial_{\mathbf{b}}^{p+1}F(z)=\frac{1}{g_0(z)}\left(\partial_{\mathbf{b}}h(z)-\sum_{j=0}^{p}\partial_{\mathbf{b}}g_j(z)\cdot\partial_{\mathbf{b}}^{p-j}F(z)-\sum_{j=1}^{p}g_j(z)\partial_{\mathbf{b}}^{p-j+1}F(z)\right). \tag{3.9}$$

Putting in the first condition of this theorem $m_0=1$, with (3.9) in view of the second condition we obtain

$$\begin{gathered}\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|\le\frac{1}{|g_0(z)|}\left(ML(z)\sum_{j=0}^{p}|g_j(z)|\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\sum_{j=0}^{p}|\partial_{\mathbf{b}}g_j(z)|\times\right.\\ \left.\times\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\sum_{j=1}^{p}|g_j(z)|\left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right|\right)\le M\sum_{j=0}^{p}\frac{|g_j(z)|}{L^j(z)|g_0(z)|}\times\\ \times L^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\sum_{j=0}^{p}|\partial_{\mathbf{b}}g_j(z)|\frac{1}{|g_0(z)|L^{j+1}(z)|}L^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\\ +\sum_{j=1}^{p}\frac{|g_j(z)|}{|g_0(z)|}\frac{1}{L^j(z)}L^j(z)\left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right|\le M\sum_{j=0}^{p}m_jL^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\\ +\sum_{j=0}^{p}M_jL^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\sum_{j=1}^{p}m_jL^j(z)\left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right|\le\\ \le M^*\left((M+1)\sum_{j=0}^{p}L^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\sum_{j=1}^{p}L^j(z)\left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right|\right)=\\ =M^*\left((M+1)\sum_{j=0}^{p}L^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\sum_{j=0}^{p-1}L^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|+\right.\\ \left.+L^{p+1}(z)|F(z)|\right)\le M^*\left((M+2)\sum_{j=0}^{p}L^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|\right).\end{gathered}$$

We divide the obtained inequality by $(p+1)!L^{p+1}(z)$

$$\begin{gathered}\frac{1}{(p+1)!L^{p+1}(z)}\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|\le M^*(M+2)\sum_{j=0}^{p}\frac{1}{(p-j)!L^{p-j}(z)}\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|\times\\ \times\frac{(p-j)!}{(p+1)!}\le\frac{2M^*(M^*+2)}{(p+1)}\max\left\{\frac{1}{k!L^k(z)}\left|\partial_{\mathbf{b}}^{k}F(z)\right|:0\le k\le p\right\},\end{gathered}$$

because

$$\sum_{j=0}^{p}\frac{(p-j)!}{(p+1)!}\leq\frac{0!+1!+2!+3!+\cdots+p!}{(p+1)!}=\frac{2\cdot 1!+2!+3!+4!+\cdots+p!}{(p+1)!}=$$
$$=\frac{2\cdot 2!+2!+3!+4!+\cdots+p!}{(p+1)!}\leq\frac{2\cdot 3!+4!+5!+\cdots+p!}{(p+1)!}\leq$$
$$\leq\frac{2\cdot 4!+5!+\cdots+p!}{(p+1)!}\leq\frac{2\cdot 5!+\cdots p!}{(p+1)!}\leq\frac{2p!}{(p+1)!}=\frac{2}{p+1}. \tag{3.10}$$

Hence, by Corollary 1.3 the function F has bounded L-index in the direction $\mathbf{b}$, because

$$\frac{1}{L^{p+1}(z)}\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|\leq M^*(M+2)\sum_{j=0}^{p}\frac{1}{L^{p-j}(z)}\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|\leq$$
$$\leq 2M^*(M+2)\max\left\{\frac{1}{L^k(z)}\left|\partial_{\mathbf{b}}^{k}F(z)\right|:0\leq k\leq p\right\}.$$

And by Lemma 2.2 corresponding estimate (3.7) holds. □

Theorem 3.4. *Let $L\in QW_{\mathbf{b}}(\mathbb{B}^n)$, functions g_0, $g_1, .., g_p$ be analytic in the unit ball and there exists $R\in[0,1)$ such that for all $z\in\mathbb{B}^n$, $|z|\geq R$, one has*

$$|g_j(z)|\leq m_jL^j(z)|g_0(z)|\ \textit{for}\ 1\leq j\leq p,$$

where m_j are some nonnegative constants. If an analytic function $F:\mathbb{B}^n\to\mathbb{C}$ satisfies equation (3.1) with $h(z)\equiv 0$ and $\forall z\in\mathbb{B}^n$, $|z|<R$, $F(z+t\mathbf{b})\not\equiv 0$ then $F(z)$ is of bounded L-index in the direction $\mathbf{b}$ and for all $z^0\in\mathbb{B}^n$, $\theta\in[0,2\pi]$

$$\overline{\lim_{|z^0+re^{i\theta}\mathbf{b}|\to 1}}\frac{\ln|F(z^0+e^{i\theta}r\mathbf{b})|}{\int_0^r L(z^0+te^{i\theta}\mathbf{b})dt}\leq\max\{p,2M^*\}, \tag{3.11}$$

where $M^*=\max\{1,m_j\}$.

Proof. The proof of this theorem is similar to the proofs of Theorems 3.2 and 3.3. In particular, from equation (3.1) with $h(z)\equiv 0$ it follows that

$$g_0(z)\partial_{\mathbf{b}}^{p}F(z)=-\left(\sum_{j=1}^{p}g_j(z)\partial_{\mathbf{b}}^{p-j}F(z)\right),$$

then

$$|g_0(z)|\ |\partial_{\mathbf{b}}^{p}F(z)|\leq\sum_{j=1}^{p}|g_j(z)|\ \left|\partial_{\mathbf{b}}^{p-j}F(z)\right|. \tag{3.12}$$

Dividing the obtained inequality by $|g_0(z)|L^p(z)$ and using the conditions of this theorem for functions $g_j(z)$, we obtain

$$\frac{1}{L^p(z)}|\partial_{\mathbf{b}}^p F(z)| \leq \sum_{j=1}^{p}\left|\frac{g_j(z)}{g_0(z)}\right|\frac{1}{L^p(z)}\left|\partial_{\mathbf{b}}^{p-j}F(z)\right| \leq \sum_{j=1}^{p}\frac{m_j}{L^{p-j}(z)}\times$$
$$\times\left|\partial_{\mathbf{b}}^{p-j}F(z)\right| \leq \sum_{j=1}^{p} m_j \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{L^k(z)} : 0\leq k\leq p-1\right\}.$$

Thus, by Corollary 1.3 the function F is of bounded L-index in the direction $\mathbf{b}$. We show that conditions of Lemma 2.2 are satisfied. Dividing inequality (3.12) by $p!L^p(z)$, we obtain

$$\frac{|\partial_{\mathbf{b}}^p F(z)|}{p!L^p(z)} \leq \sum_{j=1}^{p}\left|\frac{g_j(z)}{g_0(z)}\right|\frac{1}{p!L^p(z)}\left|\partial_{\mathbf{b}}^{p-j}F(z)\right| \leq \sum_{j=1}^{p}\frac{m_j}{L^{p-j}(z)}\frac{(p-j)!}{p!}\frac{1}{(p-j)!}\times$$
$$\times\left|\partial_{\mathbf{b}}^{p-j}F(z)\right| \leq \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)} : 0\leq k\leq p-1\right\}\sum_{j=1}^{p} m_j\frac{(p-j)!}{p!} \leq$$
$$\leq M^*\sum_{j=1}^{p}\frac{(p-j)!}{p!}\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)} : 0\leq k\leq p-1\right\} \leq$$
$$\leq \frac{2M^*}{p}\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{K!L^k(z)} : 0\leq k\leq p-1\right\}.$$

In the proof of this estimate we used an inequality (3.10), which was obtained in the proof of Theorem 3.3. Thus, by Lemma 2.2 the corresponding estimate (3.11) holds. □

Remark 3.5. The conditions in Theorems 3.1-3.4 imposed by the coefficients of equations are easy satisfied because there is some freedom to choose a function L. In the worst case we can choose the function L as an iterated exponential function in this form $A\exp_k(\frac{1}{1-|z|})$, where $\exp_k(t)=\exp(\exp_{k-1}(t))$, $A>0$ is sufficiently big number and k is integer number depending on the growth of coefficients $g_j(z)$ and $h(z)$. For example, in Theorem 3.1 this number k can be chosen such that $(|g_j(z)|/|g_0(z)|)^{1/j} = O(\exp_k(\frac{1}{1-|z|})$ for $1\leq j\leq p$, $(|\partial_{\mathbf{b}}g_j(z)|/|g_0(z)|)^{1/(j+1)} = O(\exp_k(\frac{1}{1-|z|})$ for $0\leq j\leq p$ and $|\partial_{\mathbf{b}}h(z)|/|h(z)| = O(\exp_k(\frac{1}{1-|z|})$ as $|z|\to 1-0$.

Example 3.6. Let us consider the following third order partial differential equation from [7]:

$$\partial_{\mathbf{b}}^3 F = 2(\pi b_1 z_2+\pi b_2 z_1)\partial_{\mathbf{b}}^2 F+$$
$$+2\left(\frac{(\pi b_1 z_2+\pi b_2 z_1)^2}{\cos^2\pi z_1 z_2}+2\pi b_1 b_2\tan\pi z_1 z_2\right)\partial_{\mathbf{b}}F-\frac{4\pi b_1 b_2(\pi b_1 z_2+\pi b_2 z_1)}{\cos^2\pi z_1 z_2}F \qquad (3.13)$$

where $\mathbf{b} = (b_1, b_2) \in \mathbb{C}^2 \setminus \{0\}$. It is easy to check that conditions of Theorem 3.2 are satisfied for this equation and for the functions

$$L(z_1, z_2) = \frac{|b_1 z_2 + b_2 z_1| + 1}{(1 - |z|)|\frac{1}{2} - z_1 z_2|}$$

in the unit ball. Therefore, by Theorem 3.2 every analytic solution of equation (3.13) has bounded L-index in the direction $\mathbf{b}$ and its growth is described by estimate (3.6). Namely, the function $F(z_1, z_2) = \tan(\pi z_1 z_2)$ has the bounded L-index in this direction because the function F is analytic solution in $\mathbb{B}^2$ of equation (3.13).

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A $p(x)$-Kirchhoff type problem involving the $p(x)$-Laplacian-like operators with Dirichlet boundary condition

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Abstract. This paper deals with a class of $p(x)$-Kirchhoff type problems involving the $p(x)$-Laplacian-like operators, arising from the capillarity phenomena, depending on two real parameters with Dirichlet boundary conditions. Using a topological degree for a class of demicontinuous operators of generalized (S_+), we prove the existence of weak solutions of this problem. Our results extend and generalize several corresponding results from the existing literature.

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1. Introduction

The study of differential equations and variational problems with nonlinearities and nonstandard $p(x)$-growth conditions or nonstandard $(p(x), q(x))-$ growth conditions have received a lot of attention. Perhaps the impulse for this comes from the new search field that reflects a new type of physical phenomenon is a class of nonlinear problems with variable exponents (see [26]). The motivation for this research comes from the application of similar models in physics to represent the behavior of elasticity [34] and electrorheological fluids (see [30, 32]), which have the ability to modify their mechanical properties when exposed to an electric field (see [3, 4, 7, 11, 15, 27, 28, 29]), specifically the phenomenon of capillarity, which depends on solid-liquid interfacial characteristics as surface tension, contact angle, and solid surface geometry.

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Let Ω be a bounded domain in $\mathbb{R}^N (N > 1)$ with smooth boundary denoted by $\partial\Omega$, $a \in L^\infty(\Omega)$, $p(x), k(x) \in C_+(\overline{\Omega})$, and let μ and λ be two real parameters.

In this article, we consider a class of $p(x)$-Kirchhoff type problems involving the $p(x)$-Laplacian-like operators, originated from a capillarity phenomena, depending on two real parameters with Dirichlet boundary conditions of the following form:

$$\begin{cases} -\mathcal{M}\Big(\mathcal{C}(u)\Big)\Big(\Delta^{\mathcal{L}}_{p(x)} u - |u|^{p(x)-2}u\Big) + a(x)|u|^{k(x)-2}u \\ \qquad\qquad\qquad\qquad = \mu\, g(x,u) + \lambda\, f(x,u,\nabla u) \quad \text{in } \Omega, \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\mathcal{C}(u) := \int_\Omega \frac{|\nabla u|^{p(x)} + \sqrt{1 + |\nabla u|^{2p(x)}} + |u|^{p(x)}}{p(x)} dx,$$

and

$$\Delta^{\mathcal{L}}_{p(x)} u := \operatorname{div}\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1 + |\nabla u|^{2p(x)}}}\Big)$$

is the $p(x)$-Laplacian-like operators, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions that satisfy the assumption of growth, and
$\mathcal{M} : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function.

Problems related to (1.1) have been studied by many scholars, for example, Ni and Serrin [20, 21] considered the following equation

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\Big) = f(u) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

The operator $-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\Big)$ is most often denoted by the specified mean curvature operator and $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ is the Kirchhoff stress term.

"Elliptic boundary value problems" involving the mean curvature operator play apivotal role in the mathematical analysis of several physical or geometrical issues, such as capillarity phenomena for incompressible or compressible fluids, mathematical models in physiology or in electrostatics, flux-limited diffusion phenomena, prescribed mean curvature problems for Cartesian surfaces in the Euclidean space: relevant references on these topics include [8, 9, 13, 14].

In the case when $\mathcal{M}\Big(\mathcal{C}(u)\Big) \equiv 1$, $\mu = a = 0$, $\lambda > 0$, f independent of ∇u and without the term $|u|^{p(x)-2}u$, we know that the problem (1.1) has a nontrivial solutions from [31].

For $\mathcal{M}\Big(\mathcal{C}(u)\Big) \equiv 1$, $k(x) = p(x)$, $\mu \geq 0$, $\lambda > 0$, $a \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega a > 0$ and f independent of ∇u, Afrouzi et al. [5] established some new sufficient conditions underwhich the problem (1.1), under Neumann boundary condition, possesses infinitely many weak solutions. Their discussion is based on a fully variational method and the main tool is a general critical point theorem.

Note that, in the case when

$$\mathcal{C}(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx, \;\; \Delta^{\mathcal{L}}_{p(x)} u = \Delta_{p(x)} u = \operatorname{div}\Big(|\nabla u|^{p(x)-2} \nabla u\Big),$$

$\mu = a = 0$, $\lambda = 1$, f independent of ∇u and without the term $|u|^{p(x)-2}u$, then we obtain the following problem

$$\begin{cases} -\mathcal{M}\Big(\displaystyle\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx\Big)\Delta_{p(x)} u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

which is called the $p(x)$-Kirchhoff type problem. In this case, Dai et al. [10], by a direct variational approach, established conditions ensuring the existence and multiplicity of solutions to (1.3). Furthermore, the problem (1.3) is a generalization of the stationary problem of a model introduced by Kirchhoff [17] of the following form:

$$\rho \frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.4}$$

where ρ, ρ_0, h, E, L are all constants, which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration.

Lapa et al. [19] showed, by using a Fredholm-type result for a couple of nonlinear operators, and the theory of variable exponent Sobolev spaces, the existence of weak solutions for the problem (1.1), under no-flux boundary conditions, in the case when $\mu = a = 0$, $\lambda = 1$ and f independent of ∇u.

In the present paper, we will generalize these works, by proving the existence of a weak solutions for the problem (1.1). Note that the problem (1.1) has not a variational structure, so the most usual variational methods can not used to study it. To attack it we will employ a topological degree for a class of demicontinuous operators of generalized (S_+) type of [6].

2. Preliminaries

In the analysis of problem (1.1), we will use the theory of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. For convenience, we only recall some basic facts with will be used later, we refer to [12, 18, 22, 25, 23, 24] for more details.

Let Ω be a smooth bounded domain in $\mathbb{R}^N (N > 1)$, with a Lipschitz boundary denoted by $\partial\Omega$. Set

$$C_+(\overline{\Omega}) = \Big\{p : p \in C(\overline{\Omega}) \text{ such that } p(x) > 1 \text{ for any } x \in \overline{\Omega}\Big\}.$$

For each $p \in C_+(\overline{\Omega})$, we define

$$p^+ := \max\Big\{p(x), \; x \in \overline{\Omega}\Big\} \;\text{ and }\; p^- := \min\Big\{p(x), \; x \in \overline{\Omega}\Big\}.$$

For every $p \in C_+(\overline{\Omega})$, we define

$$L^{p(x)}(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \ \text{ is measurable such that } \int_\Omega |u(x)|^{p(x)} dx < +\infty \Big\},$$

equipped with the Luxemburg norm

$$|u|_{p(x)} = \inf \Big\{ \lambda > 0 : \rho_{p(x)}\Big(\frac{u}{\lambda}\Big) \leq 1 \Big\},$$

where

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx, \ \forall\, u \in L^{p(x)}(\Omega).$$

Proposition 2.1. [12] *Let (u_n) and $u \in L^{p(x)}(\Omega)$, then*

$$|u|_{p(x)} < 1\Big(resp. = 1; > 1\Big) \ \Leftrightarrow \ \rho_{p(x)}(u) < 1\Big(resp. = 1; > 1\Big), \tag{2.1}$$

$$|u|_{p(x)} > 1 \ \Rightarrow \ |u|^{p^-}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^+}_{p(x)}, \tag{2.2}$$

$$|u|_{p(x)} < 1 \ \Rightarrow \ |u|^{p^+}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)}, \tag{2.3}$$

$$\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \ \Leftrightarrow \ \lim_{n\to\infty} \rho_{p(x)}\Big(u_n - u\Big) = 0. \tag{2.4}$$

Remark 2.2. According to (2.2) and (2.3), we have

$$|u|_{p(x)} \leq \rho_{p(x)}(u) + 1, \tag{2.5}$$

$$\rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)} + |u|^{p^+}_{p(x)}. \tag{2.6}$$

Proposition 2.3. [18] *The space $\Big(L^{p(x)}(\Omega), |\cdot|_{p(x)}\Big)$ is a separable and reflexive Banach spaces.*

Proposition 2.4. [18] *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ where*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$$

for all $x \in \Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the following Hölder-type inequality

$$\Big| \int_\Omega uv\ dx \Big| \leq \Big(\frac{1}{p-} + \frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}. \tag{2.7}$$

Remark 2.5. If p_1, $p_2 \in C_+(\overline{\Omega})$ with $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

Now, let $p \in C_+(\overline{\Omega})$ and we define $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega) = \Big\{ u \in L^{p(x)}(\Omega) \text{ such that } |\nabla u| \in L^{p(x)}(\Omega) \Big\},$$

equipped with the norm

$$||u|| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

We also define $W^{1,p(x)}_0(\Omega)$ as the subspace of $W^{1,p(x)}(\Omega)$, which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $||\cdot||$.

Proposition 2.6. [12] *If the exponent $p(x)$ satisfies the log-Hölder continuity condition, i.e. there is a constant $a > 0$ such that for every $x,\ y \in \Omega,\ x \neq y$ with $|x - y| \leq \dfrac{1}{2}$ one has*

$$|p(x) - p(y)| \leq \frac{a}{-\log|x - y|}, \tag{2.8}$$

then we have the Poincaré inequality, i.e. there exists a constant $C > 0$ depending only on Ω and the function p such that

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)},\ \forall\ u \in W_0^{1,p(x)}(\Omega). \tag{2.9}$$

In this paper we will use the following equivalent norm on $W_0^{1,p(x)}(\Omega)$

$$|u|_{1,p(x)} = |\nabla u|_{p(x)},$$

which is equivalent to $||\cdot||$.
Furthermore, we have the compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$(see [18]).

Proposition 2.7. [12, 18] *The spaces $\Big(W_0^{1,p(x)}(\Omega), |\cdot|_{1,p(x)}\Big)$ and $\Big(W_0^{1,p(x)}(\Omega), |\cdot|_{1,p(x)}\Big)$ are separable and reflexive Banach spaces.*

Remark 2.8. The dual space of $W_0^{1,p(x)}(\Omega)$ denoted $W^{-1,p'(x)}(\Omega)$, is equipped with the norm

$$|u|_{-1,p'(x)} = \inf\Big\{|u_0|_{p'(x)} + \sum_{i=1}^{N} |u_i|_{p'(x)}\Big\},$$

where the infinimum is taken on all possible decompositions $u = u_0 - \mathrm{div}F$ with $u_0 \in L^{p'(x)}(\Omega)$ and $F = (u_1, \ldots, u_N) \in (L^{p'(x)}(\Omega))^N$.

3. A review on the topological degree theory

Now, we give some results and properties from the theory of topological degree. The readers can find more information about the history of this theory in [1, 2, 6, 16].

In what follows, let X be a real separable reflexive Banach space and X^* be its dual space with dual pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ and given a nonempty subset Ω of X. Strong (weak) convergence is represented by the symbol $\to$ ($\rightharpoonup$).

Definition 3.1. Let Y be real Banach space. A operator $F : \Omega \subset X \to Y$ is said to be:

1. bounded, if it takes any bounded set into a bounded set.
2. demicontinuous, if for any sequence $(u_n) \subset \Omega,\ u_n \to u$ implies that $F(u_n) \rightharpoonup F(u)$.
3. compact, if it is continuous and the image of any bounded set is relatively compact.

Definition 3.2. A mapping $F : \Omega \subset X \to X^*$ is said to be:

1. of type (S_+), if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $\limsup\limits_{n\to\infty}\langle Fu_n, u_n - u\rangle \leq 0$, we have $u_n \to u$.

2. quasimonotone, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, we have $\limsup_{n\to\infty} \langle Fu_n,\ u_n - u\rangle \geq 0$.

Definition 3.3. Let $T : \Omega_1 \subset X \to X^*$ be a bounded operator such that $\Omega \subset \Omega_1$. For any operator $F : \Omega \subset X \to X$, we say that

1. F of type $(S_+)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup_{n\to\infty} \langle Fu_n,\ y_n - y\rangle \leq 0$, we have $u_n \to u$.
2. F has the property $(QM)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$, we have $\limsup_{n\to\infty} \langle Fu_n, y - y_n\rangle \geq 0$.

In the sequel, we consider the following classes of operators:

$$\mathcal{F}_1(\Omega) := \Big\{F : \Omega \to X^* : F \text{ is bounded, demicontinuous and of type } (S_+)\Big\},$$

$$\mathcal{F}_{T,B}(\Omega) := \Big\{F : \Omega \to X : F \text{ is bounded, demicontinuous and of type } (S_+)_T\Big\},$$

$$\mathcal{F}_T(\Omega) := \Big\{F : \Omega \to X : F \text{ is demicontinuous and of type } (S_+)_T\Big\},$$

for any $\Omega \subset D(F)$, where $D(F)$ denotes the domain of F, and any $T \in \mathcal{F}_1(\Omega)$. Now, let $\mathcal{O}$ be the collection of all bounded open sets in X and we define

$$\mathcal{F}(X) := \Big\{F \in \mathcal{F}_T(\overline{E}) :\ E \in \mathcal{O},\ \mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})\Big\},$$

where, $\mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})$ is called an essential inner map to F.

Lemma 3.4. [16, Lemma 2.3] *Let $T \in \mathcal{F}_1(\overline{E})$ be continuous and $S : D(S) \subset X^* \to X$ be demicontinuous such that $T(\overline{E}) \subset D(S)$, where E is a bounded open set in a real reflexive Banach space X. Then the following statements are true:*

1. *If S is quasimonotone, then $I + S \circ T \in \mathcal{F}_T(\overline{E})$, where I denotes the identity operator.*
2. *If S is of type (S_+), then $S \circ T \in \mathcal{F}_T(\overline{E})$.*

Definition 3.5. Suppose that E is bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\overline{E})$ is continuous and $F, S \in \mathcal{F}_T(\overline{E})$. The affine homotopy $\mathcal{H} : [0,1] \times \overline{E} \to X$ defined by

$$\mathcal{H}(t,u) := (1-t)Fu + tSu, \quad \text{for all} \quad (t,u) \in [0,1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 3.6. [16, Lemma 2.5] The above affine homotopy is of type $(S_+)_T$.

Next, as in [16] we give the topological degree for the type $\mathcal{F}(X)$.

Theorem 3.7. *Let*

$$M = \Big\{(F, E, h) : E \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{E}),\ F \in \mathcal{F}_{T,B}(\overline{E}),\ h \notin F(\partial E)\Big\},$$

then, there exists a unique degree function $d : M \longrightarrow \mathbb{Z}$ that satisfies the following properties:

1. (Normalization) *For any* $h \in E$, *we have* $d(I, E, h) = 1$.
2. (Homotopy invariance) *If* $\mathcal{H} : [0,1] \times \overline{E} \to X$ *is a bounded admissible affine homotopy with a common continuous essential inner map and* $h : [0,1] \to X$ *is a continuous path in* X *such that* $h(t) \notin \mathcal{H}(t, \partial E)$ *for all* $t \in [0,1]$, *then*

$$d(\mathcal{H}(t,\cdot), E, h(t)) = const \text{ for all } t \in [0,1].$$

3. (Existence) *If* $d(F, E, h) \neq 0$, *then the equation* $Fu = h$ *has a solution in* E.

Definition 3.8. [16, Definition 3.3] The above degree is defined as follows:

$$d(F, E, h) \ := d_B(F|_{\overline{E}_0}, E_0, h),$$

where d_B is the Berkovits degree [6] *and* E_0 *is any open subset of* E *with* $F^{-1}(h) \subset E_0$ *and* F *is bounded on* $\overline{E}_0$.

4. Existence of weak solution

In this section, we will discuss the existence of weak solutions of (1.1).

We assume that $\Omega \subset \mathbb{R}^N (N > 1)$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (2.8), $a \in L^\infty(\Omega)$, $k \in C_+(\overline{\Omega})$ with $1 < k^- \leq k(x) \leq k^+ < p^-$, $\mathcal{M} : \mathbb{R}^+ \to \mathbb{R}^+$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are functions such that:

(A_1). f is a Carathéodory function.

(A_2). There exists $\varrho > 0$ and $\gamma \in L^{p'(x)}(\Omega)$ such that

$$|f(x,\zeta,\xi)| \leq \varrho(\gamma(x) + |\zeta|^{q(x)-1} + |\xi|^{q(x)-1}).$$

(A_3). g is a Carathéodory function.

(A_4). There are $\sigma > 0$ and $\nu \in L^{p'(x)}(\Omega)$ such that

$$|g(x,\zeta)| \leq \sigma(\nu(x) + |\zeta|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and all $(\zeta, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $q, s \in C_+(\overline{\Omega})$ with $1 < q^- \leq q(x) \leq q^+ < p^-$ and $1 < s^- \leq s(x) \leq s^+ < p^-$.

(M_0). $\mathcal{M} : [0, +\infty) \to (m_0, +\infty)$ is a continuous and increasing function with $m_0 > 0$.

Remark 4.1. • Note that, for all $u, v \in W_0^{1,p(x)}(\Omega)$

$$\mathcal{M}\Big(\mathcal{C}(u)\Big) \int_\Omega \Big(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1 + |\nabla u|^{2p(x)}}}\Big)\nabla v + |u|^{p(x)-2}u\, v\Big)dx$$

is well defined (see [19]).

- $a(x)|u|^{k(x)-2}u$, $\mu\, g(x,u)$ and $\lambda\, f(x, u, \nabla u)$ are belongs to $L^{p'(x)}(\Omega)$ under $u \in W_0^{1,p(x)}(\Omega)$, the assumptions (A_2) and (A_4) and the given hypotheses about the exponents p, k, q and s because:

$$\gamma \in L^{p'(x)}(\Omega),\ \nu \in L^{p'(x)}(\Omega),\ r(x) = (q(x) - 1)p'(x) \in C_+(\overline{\Omega})$$

with $r(x) < p(x)$, $\beta(x) = (k(x)-1)p'(x) \in C_+(\overline{\Omega})$ with $\beta(x) < p(x)$ and

$$\kappa(x) = (s(x)-1)p'(x) \in C_+(\overline{\Omega}) \text{ with } \kappa(x) < p(x).$$

Then, by Remark 2.5 we can conclude that

$$L^{p(x)} \hookrightarrow L^{r(x)},\ L^{p(x)} \hookrightarrow L^{\beta(x)} \text{ and } L^{p(x)} \hookrightarrow L^{\kappa(x)}.$$

Hence, since $v \in L^{p(x)}(\Omega)$, we have

$$\Big(-a(x)|u|^{k(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big)v \in L^1(\Omega).$$

This implies that, the integral

$$\int_\Omega \Big(-a(x)|u|^{k(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big)v dx$$

exists.

Then, we shall use the definition of weak solution for problem (1.1) in the following sense:

Definition 4.2. We say that a function $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1.1), if for any $v \in W_0^{1,p(x)}(\Omega)$, it satisfies the following:

$$\begin{aligned}\mathcal{M}\Big(\mathcal{C}(u)\Big)\int_\Omega \bigg(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla v + |u|^{p(x)-2}u\, v\bigg)dx \\ = \int_\Omega \Big(-a(x)|u|^{k(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big)v dx.\end{aligned}$$

Before giving our main result we first give two results that will be used later.

Lemma 4.3. *If (M_0) holds, then the operator $\mathcal{T} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by*

$$\langle \mathcal{T}u, v\rangle = \mathcal{M}\Big(\mathcal{C}(u)\Big)\int_\Omega \bigg(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla v + |u|^{p(x)-2}u\, v\bigg)dx,$$

is continuous, bounded, strictly monotone and is of type (S_+).

Proof. Let us consider the following functional:

$$\mathcal{J}(u) := \widehat{\mathcal{M}}\Big(\mathcal{C}(u)\Big), \quad \text{where} \quad \widehat{\mathcal{M}}(s) = \int_0^s \mathcal{M}(\tau)\mathrm{d}\tau,$$

such that $\mathcal{M}(\tau)$ satisfies the assumption (M_0).
From [19], it is obvious that $\mathcal{J}$ is a continuously Gâteaux differentiable function whose Gâteaux derivative at the point $u \in W_0^{1,p(x)}(\Omega)$ is the functional $\mathcal{T}(u) := \mathcal{J}'(u) \in W^{-1,p'(x)}(\Omega)$ given by

$$\langle \mathcal{T}u, v\rangle = \mathcal{M}\Big(\mathcal{C}(u)\Big)\int_\Omega \bigg(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla v + |u|^{p(x)-2}u\, v\bigg)dx,$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$ where $\langle \cdot,\cdot\rangle$ means the duality pairing between $W^{-1,p'(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$.

Hence, by using the similar argument as in the Theorem 3.1. of [19] and in the Proposition 3.1. of [31], we conclude that $\mathcal{T}$ is continuous, bounded, strictly monotone and is of type (S_+). $\square$

Proposition 4.4. *Assume that the assumptions $(A_1)-(A_4)$ hold, then the operator*

$$\mathcal{S} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$$
$$\langle \mathcal{S}u, v\rangle = -\int_\Omega \Big(- a(x)|u|^{k(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big) v dx,$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$, is compact.

Proof. In order to prove this proposition, we proceed in four steps.
Step 1: Let $\Psi_1 : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be an operator defined by

$$\Psi_1 u(x) := -\mu\, g(x,u).$$

In this step, we prove that the operator Ψ_1 is bounded and continuous.
First, let $u \in W_0^{1,p(x)}(\Omega)$, bearing (A_4) in mind and using (2.5) and (2.6), we infer

$$\begin{aligned}
|\Psi_1 u|_{p'(x)} &\le \rho_{p'(x)}(\Psi_1 u) + 1 \\
&= \int_\Omega |\mu\, g(x,u(x))|^{p'(x)} dx + 1 \\
&= \int_\Omega |\mu|^{p'(x)} |g(x,u(x)|^{p'(x)} dx + 1 \\
&\le \Big(|\mu|^{p'^-} + |\mu|^{p'^+}\Big) \int_\Omega |\sigma\Big(\nu(x) + |u|^{s(x)-1}\Big)|^{p'(x)} dx + 1 \\
&\le const\Big(|\mu|^{p'^-} + |\mu|^{p'^+}\Big) \int_\Omega \Big(|\nu(x)|^{p'(x)} + |u|^{\kappa(x)}\Big) dx + 1 \\
&\le const\Big(|\mu|^{p'^-} + |\mu|^{p'^+}\Big) \Big(\rho_{p'(x)}(\nu) + \rho_{\kappa(x)}(u)\Big) + 1 \\
&\le const\Big(|\nu|_{p(x)}^{p'^+} + |u|_{\kappa(x)}^{\kappa^+} + |u|_{\kappa(x)}^{\kappa^-}\Big) + 1.
\end{aligned}$$

Then, we deduce from (2.9) and $L^{p(x)} \hookrightarrow L^{\kappa(x)}$, that

$$|\Psi_1 u|_{p'(x)} \le const\Big(|\nu|_{p(x)}^{p'^+} + |u|_{1,p(x)}^{\kappa^+} + |u|_{1,p(x)}^{\kappa^-}\Big) + 1,$$

that means Ψ_1 is bounded on $W_0^{1,p(x)}(\Omega)$.
Second, we show that the operator Ψ_1 is continuous. To this purpose let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. We need to show that $\Psi_1 u_n \to \Psi_1 u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.
Note that if $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$. Hence there exist a subsequence (u_k) of (u_n) and ϕ in $L^{p(x)}(\Omega)$ such that

$$u_k(x) \to u(x) \quad \text{and} \quad |u_k(x)| \le \phi(x), \tag{4.1}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.
Hence, from (A_2) and (4.1), we have

$$|g(x, u_k(x))| \le \sigma(\nu(x) + |\phi(x)|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
On the other hand, thanks to (A_3) and (4.1), we get, as $k \longrightarrow \infty$

$$g(x, u_k(x)) \to g(x, u(x)) \ \text{ a.e. } \ x \in \Omega.$$

Seeing that

$$\nu + |\phi|^{s(x)-1} \in L^{p'(x)}(\Omega) \ \text{ and } \ \rho_{p'(x)}(\Psi_1 u_k - \Psi_1 u) = \int_\Omega |g(x, u_k(x)) - g(x, u(x))|^{p'(x)} dx,$$

then, from the Lebesgue's theorem and the equivalence (2.4), we have

$$\Psi_1 u_k \to \Psi_1 u \ \text{ in } \ L^{p'(x)}(\Omega),$$

and consequently

$$\Psi_1 u_n \to \Psi_1 u \ \text{ in } \ L^{p'(x)}(\Omega),$$

that is, Ψ_1 is continuous.
Step 2: We define the operator $\Psi_2 : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Psi_2 u(x) := a(x)|u(x)|^{k(x)-2}u(x).$$

We will prove that Ψ_2 is bounded and continuous.
It is clear that Ψ_2 is continuous. Next we show that Ψ_2 is bounded.
Let $u \in W_0^{1,p(x)}(\Omega)$ and using (2.5) and (2.6), we obtain

$$\begin{aligned}
|\Psi_2 u|_{p'(x)} &\le \rho_{p'(x)}(\Psi_2 u) + 1 \\
&= \int_\Omega |a(x)|u|^{k(x)-2}u|^{p'(x)} dx + 1 \\
&= \int_\Omega |a(x)|^{p'(x)}|u|^{(k(x)-1)p'(x)} dx + 1 \\
&\le ||a||^{p'}_{L^\infty(\Omega)} \int_\Omega |u|^{\beta(x)} dx + 1 \\
&= ||a||^{p'}_{L^\infty(\Omega)} \rho_{\beta(x)}(u) + 1 \\
&\le ||a||^{p'}_{L^\infty(\Omega)} \Big(|u|^{\beta^-}_{\beta(x)} + |u|^{\beta^+}_{\beta(x)} \Big) + 1.
\end{aligned}$$

Hence, we deduce from $L^{p(x)} \hookrightarrow L^{\beta(x)}$ and (2.9) that

$$|\Psi_2 u|_{p'(x)} \le const \Big(|u|^{\beta^-}_{1,p(x)} + |u|^{\beta^+}_{1,p(x)} \Big) + 1,$$

and consequently, Ψ_2 is bounded on $W_0^{1,p(x)}(\Omega)$.
Step 3: Let us define the operator $\Psi_3 : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Psi_3 u(x) := -\lambda\, f(x, u(x), \nabla u(x)).$$

We will show that Ψ_3 is bounded and continuous.
Let $u \in W_0^{1,p(x)}(\Omega)$. According to (A_2) and the inequalities (2.5) and (2.6), we obtain

$$\begin{aligned}
|\Psi_3 u|_{p'(x)} &\leq \rho_{p'(x)}(\Psi_3 u) + 1 \\
&= \int_\Omega |\lambda f(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&= \int_\Omega |\lambda|^{p'(x)} |f(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&\leq \Big(|\lambda|^{p'^-} + |\lambda|^{p'^+}\Big) \int_\Omega |\varrho\Big(\gamma(x) + |u|^{q(x)-1} + |\nabla u|^{q(x)-1}\Big)|^{p'(x)} dx + 1 \\
&\leq const\Big(|\lambda|^{p'^-} + |\lambda|^{p'^+}\Big) \int_\Omega \Big(|\gamma(x)|^{p'(x)} + |u|^{r(x)} + |\nabla u|^{r(x)}\Big) dx + 1 \\
&\leq const\Big(|\lambda|^{p'^-} + |\lambda|^{p'^+}\Big)\Big(\rho_{p'(x)}(\gamma) + \rho_{r(x)}(u) + \rho_{r(x)}(\nabla u)\Big) + 1 \\
&\leq const\Big(|\gamma|_{p(x)}^{p'^+} + |u|_{r(x)}^{r^+} + |u|_{r(x)}^{r^-} + |\nabla u|_{r(x)}^{r^+} + |\nabla u|_{r(x)}^{r^-}\Big) + 1.
\end{aligned}$$

Taking into account that $L^{p(x)} \hookrightarrow L^{r(x)}$ and (2.9), we have then

$$|\Psi_3 u|_{p'(x)} \leq const\Big(|\gamma|_{p(x)}^{p'^+} + |u|_{1,p(x)}^{r^+} + |u|_{1,p(x)}^{r^-}\Big) + 1,$$

and consequently Ψ_3 is bounded on $W_0^{1,p(x)}(\Omega)$.
It remains to show that Ψ_3 is continuous. Let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, we need to show that $\Psi_3 u_n \to \Psi_3 u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.
Note that if $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in $(L^{p(x)}(\Omega))^N$. Hence, there exist a subsequence (u_k) and ϕ in $L^{p(x)}(\Omega)$ and ψ in $(L^{p(x)}(\Omega))^N$ such that

$$u_k(x) \to u(x) \quad \text{and} \quad \nabla u_k(x) \to \nabla u(x), \tag{4.2}$$

$$|u_k(x)| \leq \phi(x) \quad \text{and} \quad |\nabla u_k(x)| \leq |\psi(x)|, \tag{4.3}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.
Hence, thanks to (A_1) and (4.2), we get, as $k \longrightarrow \infty$

$$f(x, u_k(x), \nabla u_k(x)) \to f(x, u(x), \nabla u(x)) \quad \text{a.e. } \ x \in \Omega.$$

On the other hand, from (A_2) and (4.3), we can deduce the estimate

$$|f(x, u_k(x), \nabla u_k(x))| \leq \varrho(\gamma(x) + |\phi(x)|^{q(x)-1} + |\psi(x)|^{q(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
Seeing that

$$\gamma + |\phi|^{q(x)-1} + |\psi(x)|^{q(x)-1} \in L^{p'(x)}(\Omega),$$

and taking into account the equality

$$\rho_{p'(x)}(\Psi_3 u_k - \Psi_3 u) = \int_\Omega |f(x, u_k(x), \nabla u_k(x)) - f(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

then, we conclude from the Lebesgue's theorem and (2.4) that

$$\Psi_3 u_k \to \Psi_3 u \quad \text{in} \quad L^{p'(x)}(\Omega),$$

and consequently

$$\Psi_3 u_n \to \Psi_3 u \ \text{ in } \ L^{p'(x)}(\Omega),$$

and then Ψ_3 is continuous.

Step 4: Let $I^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be the adjoint operator of the operator $I : W_0^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$.

We then define

$$I^* \circ \Psi_1 : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

$$I^* \circ \Psi_2 : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

and

$$I^* \circ \Psi_3 : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega).$$

On another side, taking into account that I is compact, then I^* is compact. Thus, the compositions $I^* \circ \Psi_1$, $I^* \circ \Psi_2$ and $I^* \circ \Psi_3$ are compact, that means

$$\mathcal{S} = I^* \circ \Psi_1 + I^* \circ \Psi_2 + I^* \circ \Psi_3$$

is compact. With this last step the proof of Proposition 4.4 is completed. □

We are now in the position to give the existence result of weak solution for (1.1).

Theorem 4.5. *Assume that* $(A_1) - (A_4)$ *and* (M_0) *hold, then the problem* (1.1) *admits at least one weak solution* u *in* $W_0^{1,p(x)}(\Omega)$.

Proof. We will reduce the problem (1.1) to a new one governed by a Hammerstein equation, and we will apply the theory of topological degree introduced in Section 3. For all $u, v \in W_0^{1,p(x)}(\Omega)$, we define the operators $\mathcal{T}$ and $\mathcal{S}$, as defined in Lemma 4.3 and Proposition 4.4 respectively,

$$\mathcal{T} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)$$
$$\langle \mathcal{T}u, v\rangle = \mathcal{M}\Big(\mathcal{C}(u)\Big) \int_\Omega \Big(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla v + |u|^{p(x)-2}u\, v\Big)dx,$$

and

$$\mathcal{S} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)$$
$$\langle \mathcal{S}u, v\rangle = -\int_\Omega \Big(-a(x)|u|^{k(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big)v dx.$$

Consequently, the problem (1.1) is equivalent to the equation

$$\mathcal{T}u + \mathcal{S}u = 0, \quad u \in W_0^{1,p(x)}(\Omega). \tag{4.4}$$

Taking into account that, by Lemma 4.3, the operator $\mathcal{T}$ is a continuous, bounded, strictly monotone and of type (S_+), then, by [33, Theorem 26 A], the inverse operator

$$\mathcal{L} := \mathcal{T}^{-1} : W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega),$$

is also bounded, continuous, strictly monotone and of type (S_+).

On another side, according to Proposition 4.4, we have that the operator $\mathcal{S}$ is bounded, continuous and quasimonotone.

Consequently, following Zeidler's terminology [33], the equation (4.4) is equivalent to the following abstract Hammerstein equation

$$u = \mathcal{L}v \quad \text{and} \quad v + \mathcal{S} \circ \mathcal{L}v = 0, \quad u \in W_0^{1,p(x)}(\Omega) \text{ and } v \in W^{-1,p'(x)}(\Omega). \tag{4.5}$$

Seeing that (4.4) is equivalent to (4.5), then to solve (4.4) it is thus enough to solve (4.5). In order to solve (4.5), we will apply the Berkovits topological degree introduced in Section 3.
First, let us set

$$\mathcal{B} := \Big\{ v \in W^{-1,p'(x)}(\Omega) : \exists\, t \in [0,1] \text{ such that } v + t\mathcal{S} \circ \mathcal{L}v = 0 \Big\}.$$

Next, we show that $\mathcal{B}$ is bounded in $\in W^{-1,p'(x)}(\Omega)$.
Let us put $u := \mathcal{L}v$ for all $v \in \mathcal{B}$.
Taking into account that $|\mathcal{L}v|_{1,p(x)} = |\nabla u|_{p(x)}$, then we have the following two cases:
First case: If $|\nabla u|_{p(x)} \leq 1$, then $|\mathcal{L}v|_{1,p(x)} \leq 1$, that means $\Big\{ \mathcal{L}v : v \in \mathcal{B} \Big\}$ is bounded.
Second case: If $|\nabla u|_{p(x)} > 1$, then, we deduce from (2.2), (A_2) and (A_4), the inequalities (2.7) and (2.6) and the Young's inequality that

$$\begin{aligned}
&|\mathcal{L}v|_{1,p(x)}^{p^-} \\
&\leq \rho_{p(x)}(\nabla u) \\
&\leq \langle \mathcal{T}u,\ u \rangle \\
&= \langle v,\ \mathcal{L}v \rangle \\
&= -t\langle \mathcal{S} \circ \mathcal{L}v,\ \mathcal{L}v \rangle \\
&= t \int_\Omega \Big(- a(x)|u|^{k(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u) \Big) u dx \\
&\leq t \max(||a||_{L^\infty(\Omega)}, \sigma|\mu|, \varrho|\lambda|) \Big(\rho_{k(x)}(u) + \int_\Omega |\nu(x)u(x)| dx + \int_\Omega |\gamma(x)u(x)| dx \\
&\qquad\qquad + \rho_{s(x)}(u) + \rho_{q(x)}(u) + \int_\Omega |\nabla u|^{q(x)-1}|u| dx \Big) \\
&\leq \mathit{const} \Big(|u|_{k(x)}^{k^-} + |u|_{k(x)}^{k^+} + |\nu|_{p'(x)}|u|_{p(x)} + |\gamma|_{p'(x)}|u|_{p(x)} + |u|_{s(x)}^{s^+} + |u|_{s(x)}^{s^-} \\
&\qquad\qquad + |u|_{q(x)}^{q^+} + |u|_{q(x)}^{q^-} + \frac{1}{q'-}\rho_{q(x)}(\nabla u) + \frac{1}{q-}\rho_{q(x)}(u) \Big) \\
&\leq \mathit{const} \Big(|u|_{k(x)}^{k^-} + |u|_{k(x)}^{k^+} + |u|_{p(x)} + |u|_{s(x)}^{s^+} + |u|_{s(x)}^{s^-} + |u|_{q(x)}^{q^+} + |u|_{q(x)}^{q^-} + |\nabla u|_{q(x)}^{q^+} \Big),
\end{aligned}$$

then, according to $L^{p(x)} \hookrightarrow L^{k(x)}$, $L^{p(x)} \hookrightarrow L^{s(x)}$ and $L^{p(x)} \hookrightarrow L^{q(x)}$, we get

$$|\mathcal{L}v|_{1,p(x)}^{p^-} \leq \mathit{const} \Big(|\mathcal{L}v|_{1,p(x)}^{k^+} + |\mathcal{L}v|_{1,p(x)} + |\mathcal{L}v|_{1,p(x)}^{s^+} + |\mathcal{L}v|_{1,p(x)}^{q^+} \Big),$$

what implies that $\Big\{ \mathcal{L}v : v \in \mathcal{B} \Big\}$ is bounded.
On the other hand, we have that the operator is $\mathcal{S}$ is bounded, then $\mathcal{S} \circ \mathcal{L}v$ is bounded.
Thus, thanks to (4.5), we have that $\mathcal{B}$ is bounded in $W^{-1,p'(x)}(\Omega)$.

However, $\exists\ a > 0$ such that

$$|v|_{-1,p'(x)} < a \quad \text{for all} \quad v \in \mathcal{B},$$

which leads to

$$v + t\mathcal{S} \circ \mathcal{L}v \neq 0, \quad v \in \partial\mathcal{B}_a(0) \quad \text{and} \quad t \in [0,1],$$

where $\mathcal{B}_a(0)$ is the ball of center 0 and radius a in $W^{-1,p'(x)}(\Omega)$.
Moreover, by Lemma 3.4, we conclude that

$$I + \mathcal{S} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L}}(\overline{\mathcal{B}_a(0)}) \quad \text{and} \quad I = \mathcal{T} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L}}(\overline{\mathcal{B}_a(0)}).$$

On another side, taking into account that I, $\mathcal{S}$ and $\mathcal{L}$ are bounded, then $I + \mathcal{S} \circ \mathcal{L}$ is bounded. Hence, we infer that

$$I + \mathcal{S} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L},B}(\overline{\mathcal{B}_a(0)}) \quad \text{and} \quad I = \mathcal{T} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L},B}(\overline{\mathcal{B}_a(0)}).$$

Now, we define the homotopy $\mathcal{H} : [0,1] \times \overline{\mathcal{B}_a(0)} \to W^{-1,p'(x)}(\Omega)$ by

$$\mathcal{H}(t,\vartheta) := \vartheta + t\mathcal{S} \circ \mathcal{L}\vartheta.$$

Applying the homotopy invariance and normalization property of the degree d seen in Theorem 3.7, we have

$$d(I + \mathcal{S} \circ \mathcal{L}, \mathcal{B}_a(0), 0) = d(I, \mathcal{B}_a(0),\ 0) = 1 \neq 0.$$

Since $d\big(I + \mathcal{S} \circ \mathcal{L}, \mathcal{B}_a(0), 0\big) \neq 0$, then by the existence property of the degree d stated in Theorem 3.7, we conclude that there exists $\vartheta \in \mathcal{B}_a(0)$ which verifies

$$\big(I + \mathcal{S} \circ \mathcal{L}\big)(\vartheta) = 0 \Leftrightarrow \vartheta + \mathcal{S} \circ \mathcal{L}\vartheta = 0 \Leftrightarrow \mathcal{T} \circ \mathcal{L}\vartheta + \mathcal{S} \circ \mathcal{L}\vartheta = 0.$$

Hence, we conclude that $u = \mathcal{L}v$ is a weak solutions of (1.1). The proof is completed. $\square$

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Invariant regions and global existence of uniqueness weak solutions for tridiagonal reaction-diffusion systems

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Abstract. In this paper we study the existence of uniqueness global weak solutions for $m \times m$ reaction-diffusion systems for which two main properties hold: the positivity of the weak solutions and the total mass of the components are preserved with time. Moreover we suppose that the non-linearities have critical growth with respect to the gradient. The technique we use here in order to prove global existence is in the same spirit of the method developed by Boccardo, Murat, and Puel for a single equation.

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1. Introduction

In [26, 27], the authors obtained a global existence of solutions for the coupled semilinear reaction-diffusion system with diagonal by order 2, and m, triangular, and full matrix of diffusion coefficients. By combining the compact semigroup methods and some L^1 estimates, we show that global solutions exist for a large class of the term of reaction. In the works [8, 14], we find new developed methods based on truncation functions, fixed point theorems and compactness, etc to prove establish the existence of global solutions.

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In the present work we consider the problem

$$\begin{cases} \frac{\partial U}{\partial t} - A_m \Delta U = F\left(t, x, U, \nabla U\right) & \text{on }]0, +\infty[\times \Omega, \\ U = 0 \quad \text{or} \quad \frac{\partial U}{\partial \eta} = 0 & \text{on }]0, +\infty[\times \partial\Omega, \\ U\left(0, x\right) = U_0\left(x\right) & \text{on } \Omega, \end{cases} \tag{1.1}$$

by using a technique based on L^1-estimate we establish a global existence result of the solution.
We consider the m-equations of reaction-diffusion system (1.1), with $m \geq 2$, where Ω is an open bounded domain of class C^1 in $\mathbb{R}^n$, the vectors U, F, U_0 and the matrix A_m are defined as:

$$\begin{cases} U = (u_1, \ldots, u_m)^T = ((u_s)_{s=1}^m)^T, \\ \nabla U = (\nabla u_1, \ldots, \nabla u_m)^T = ((\nabla u_s)_{s=1}^m)^T, \\ F = (F_1, \ldots, F_m)^T = ((F_s)_{s=1}^m)^T, \\ U_0 = (u_1^0, \ldots, u_m^0)^T = ((u_s^0)_{s=1}^m)^T. \end{cases}$$

$$A_m = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ c_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & c_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{m-1} \\ 0 & \cdots & 0 & c_{m-1} & a_m \end{pmatrix}. \tag{1.2}$$

The nonlinearities F_s, $1 \leq s \leq m$, have critical growth with respect to $|\nabla U|$, and the constants $(a_i)_{i=1}^m$, $(b_i)_{i=1}^{m-1}$ et $(c_i)_{i=1}^{m-1}$ are supposed to be strictly positive and satisfy the condition

$$\cos^2\left(\frac{\pi}{m+1}\right) < \frac{a_i a_{i+1}}{\left(b_i + c_i\right)^2} \tag{1.3}$$

which reflects the parabolicity of the system and implies at the same time that the diffusion matrix is positive defnite. That means the eigenvalues $(\lambda_i)_{i=1}^m$, $(\lambda_1 > \lambda_2 > \ldots > \lambda_m)$, of A_m are positive.
Note that $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative on boundary $\partial\Omega$.
The initial data are assumed to be in the regions:

$$\sum\nolimits_{\mathfrak{S},\mathfrak{Z}} = \left\{ U_0 \in \mathbb{R}^m : \left\{ \begin{array}{ll} w_z^0 = \langle V_z, U_0 \rangle \leq 0 & \text{if } z \in \mathfrak{Z} \\ w_s^0 = \langle V_s, U_0 \rangle \geq 0 & \text{if } s \in \mathfrak{S} \end{array} \right\} \right\}, \tag{1.4}$$

where

$$\mathfrak{S} \cap \mathfrak{Z} = \phi, \ \mathfrak{S} \cup \mathfrak{Z} = \{1, 2, \ldots, m\}.$$

The notation $\langle ., . \rangle$ denotes the inner product in $\mathbb{R}^m$ and $V_s = (v_{s1}, \ldots, v_{sm})^T$ the eigenvector of the diffusion matrix A_m associated with the eigenvalue $(\lambda_s)_{s=1}^m$. Hence, we can see that there are 2^m regions.
This work represents a generalization to the parabolic case study did in the elliptic case (see [7]) for these systems of arbitrary order. This passage in parabolic case, needs new approaches and also technical difficulties to be overcome. We will explain in detail here.

We found a good idea to present our work as follows: we start initially with an introduction that presents the state of the art of the area studied and some recall the main results obtained previously. This will highlight the contribution of our work and its originality. In the second section we give the definition of the notion of solution used here. We then present the main results of this work. In the last section, we give the proof of global existence and uniqueness of our reaction-diffusion system. This is done in three steps: in the first we truncate the system, the latter we give suitable estimates on the approximate solutions and in the last step we show the convergence of the approximating system by using the technics introduced by Boccardo *et al.* [13] and Dall'Aglio and Orsina [15].

2. Eigenvalues and eigenvectors of the diffusion matrix

The usual norms in the spaces $L^1(\Omega)$, $L^\infty(\Omega)$ and $C\left(\overline{\Omega}\right)$ are denoted respectively by:

$$\|u\|_1 = \int_\Omega |u(x)|\,dx,$$

$$\|u\|_\infty = ess\sup_{x\in\Omega} |u(x)| \quad \text{and} \quad \|u\|_{C(\overline{\Omega})} = \max_{x\in\overline{\Omega}} |u(x)|.$$

For any initial data in $C\left(\overline{\Omega}\right)$ or $L^\infty(\Omega)$ local existence and uniqueness of solutions to the initial values problem (1.1) follow from the basic existence theory for abstract semilinear differential equations (see Friedman [16], Henry [17], Pazzy [28]).

Our aim in this section is to get a three term reccurence relation of characteristic polynomial of matrix A of dimension $m \times m$ in terms of matrices of dimensions $(m-1)\times(m-1)$ and $(m-2)\times(m-2)$ so the eigenvectors of this matrix. The solutions of characteristic polynomial $\det(A_m - \lambda I_m) = 0$ are λ which represent eigenvalues of the matrix A_m. We denote the characteristic polynomial of A_m, A_{m-1}, A_{m-2} by $\phi_m(\lambda)$, $\phi_{m-1}(\lambda)$, $\phi_{m-2}(\lambda)$ respectively.

Lemma 2.1 (See [22]). *Let A_m be the tridiagonal matrix defined in (1.2), the eigenvalues of A_m are distinct and interlace strictly with eigenvalues of A_{m-1} for $m \geq 2$. Where*

$$\phi_0(\lambda) = 1,\ \phi_1(\lambda) = a_1 - \lambda,\ \phi_m(\lambda) = (\lambda - a_m)\,\phi_{m-1}(\lambda) - b_{m-1}c_{m-1}\phi_{m-2}(\lambda). \tag{2.1}$$

Lemma 2.2 (See Andelic and Fonseca [9]). *Let A_m be the real symmetric tridiagonal matrix definied in (1.2), with diagonal entries positive.*
If

$$\cos^2\left(\frac{\pi}{m+1}\right) < \frac{a_i a_{i+1}}{(b_i + c_i)^2},\ \ for \quad i = 1, \ldots, m-1,$$

then A_m is positive definite.

We remark that general characterization in terms of the eigenvalues, i.e. A_m is positive definite if and only if all its eigenvalues are positive.

Lemma 2.3. *Let λ_s for $s = 1, \ldots, m$ be the eigenvalues of the tridiagonal matrix A_m. Then the eigenvectors $V_s = (v_{s1}, \ldots, v_{sm})^T$ associated to λ_s for $s = 1, \ldots, m$ are given by the following expressions*

$$\begin{cases} v_{sm} = 1, \\ v_{s(m-1)} = \frac{\lambda_s - a_m}{c_{m-1}}, \\ v_{s(\ell-1)} = -\frac{b_\ell v_{s(\ell+1)} + (a_\ell - \lambda_s) v_{s\ell}}{c_{\ell-1}}, \quad \ell = 2, \ldots, m-1. \end{cases} \tag{2.2}$$

Proof. Recall that the diffusion matrix is positive definite, hence its eigenvalues are necessarily positive. The eigenvectors of the diffusion matrix associated with the eigenvalues λ_s are defind as $V_s = (v_{s1}, v_{s2}, \ldots, v_{sm})^T$. For an eigenpair (λ_s, V_s), the components in $A_m V = \lambda V$ are

$$\begin{cases} a_1 v_1 + b_1 v_2 = \lambda v_1 \\ c_{\ell-1} v_{\ell-1} + a_\ell v_\ell + b_\ell v_{\ell+1} = \lambda v_\ell, \quad (2 \leq \ell \leq m-1) \\ c_{m-1} v_{m-1} + a_m v_m = \lambda v_m \end{cases}$$

if $v_m = 0$, the assumption $b_i \neq 0$, $c_i \neq 0$ for all $i = 1, \ldots, m-1$ we said that all v_{si} are zero. We can therefore take $v_m = 1$ and $(v_1, v_2, \ldots, v_{m-1})$ is a solution of upper triangular system

$$\begin{cases} c_{\ell-1} v_{\ell-1} + (a_\ell - \lambda) v_\ell + b_\ell v_{\ell+1} = 0 \quad (2 \leq \ell \leq m-2) \\ c_{m-2} v_{m-2} + (a_{m-1} - \lambda) v_{m-1} = -b_{m-1} \\ c_{m-1} v_{m-1} = \lambda - a_m \end{cases}$$

the solution of this system is given by

$$\begin{cases} v_{m-1} = \frac{\lambda - a_m}{c_{m-1}}, \\ v_{\ell-1} = -\frac{b_\ell v_{\ell+1} + (a_\ell - \lambda) v_\ell}{c_{\ell-1}}, \ (\ell = 2, \ldots, m-1). \end{cases}$$

□

3. Diagonalizing system (1.1)

Usually to construct an invariant regions for systems such (1.1) we make a linear change of variables u_i to obtain a new equivalent system with diagonal diffusion matrix for which standard techniques can be applied to deduce global existence (see [1, 2, 3, 4, 5, 21]).

Let $V_s = (v_{s1}, \ldots, v_{sm})^T$ be an eigenvector of the matrix A_m associated with its eigenvalue $(\lambda_s)_{s=1}^m$ where $\lambda_1 > \lambda_2 > \ldots > \lambda_m$. Multiplying the k^{th} equation of (1.1) by $(-1)^{i_s} V_{sk}$, $i_s = 1, 2$ and $k = 1, \ldots, m$, and adding the resulting equations, we get

$$\begin{cases} \frac{\partial W}{\partial t} - diag\left(\lambda_1, \lambda_2, \ldots, \lambda_m\right) \Delta W = \Psi\left(t, x, W, \nabla W\right) & \text{on }]0, +\infty[\times \Omega, \\ W = 0 \quad \text{or} \quad \frac{\partial W}{\partial \eta} = 0 & \text{on }]0, +\infty[\times \partial\Omega, \\ W\left(0, x\right) = W_0\left(x\right) & \text{on } \Omega, \end{cases} \tag{3.1}$$

where

$$\begin{cases} W=((w_s)_{s=1}^m)^T, \quad \nabla W=((\nabla w_s)_{s=1}^m)^T, \quad w_s=\left\langle (-1)^{i_s} V_s, U\right\rangle, \\ \Psi=((\Psi_s)_{s=1}^m)^T, \quad \Psi_s=\left\langle (-1)^{i_s} V_s, F\right\rangle, \\ W_0=((w_s^0)_{s=1}^m)^T, \qquad w_s^0=\left\langle (-1)^{i_s} V_s, U_0\right\rangle,\ m\geq 2. \end{cases}$$

for all $i_s=\{1,2\}$.

Proposition 3.1. *The system* (3.1) *admits a unique classical solution* W *on* $[0,T_{\max})\times\Omega$, *where* $T_{\max}\left(\left\|w_1^0\right\|_\infty,\left\|w_2^0\right\|_\infty,\ldots,\left\|w_m^0\right\|_\infty\right)$ *denotes the eventual blow-up time. Furthermore, if* $T_{\max}<+\infty$, *then*

$$\lim_{t\to T_{\max}}\sum_{s=1}^m \|w_s(t,.)\|_\infty=+\infty.$$

Therefore, if there exists a positive constant C *such that*

$$\sum_{s=1}^m \|w_s(t,.)\|_\infty\leq C \quad \text{for all} \quad t\in[0,T_{\max}),$$

then, $T_{\max}=+\infty$.

4. Statement of the main result

4.1. Assumptions

Let us, now introduce for w_s^0 the hypotheses, for all $1\leq s\leq m$

(A1) The initial conditions are in $\sum_{\mathfrak{S},3}$, w_s^0, are nonnegative functions in $L^1(\Omega)$.

The following assumptions are also made on the function Ψ defined by:

$$\Psi=((\Psi_s)_{s=1}^m)^T, \quad \Psi_s=\left\langle (-1)^{i_s} V_s, F\right\rangle, \quad i_s=1,2.$$

(A2) Ψ_s are continuously differentiable on $\mathbb{R}_+^m$ and Ψ_s, $s=\overline{1,m}$, are quasi-positives functions which means that, for $s=\overline{1,m}$

$$[w_1\geq 0,\ldots,w_{s-1}\geq 0, w_{s+1}\geq 0,\ldots,w_m\geq 0],$$

implies

$$\begin{cases} \Psi_s(t,x,w_1,\ldots,w_{s-1},0,w_{s+1},\ldots,w_m,p_1,\ldots,p_{s-1},0,p_{s+1},\ldots,p_m)\geq 0. \\ \quad \text{for all } 1\leq s\leq m,\ (W,p)\in(\mathbb{R}^+)^m\times\mathbb{R}^{Nm} \text{ and for a.e. } (t,x)\in Q_T \end{cases}$$

These conditions on Ψ guarantee local existence of unique, nonnegative classical solutions on a maximal time interval $[0,T_{\max})$, see Hollis and Morgan [20].

(A3) The inequality

$$\langle S,\Psi(t,x,W,\nabla W)\rangle\leq C_1(1+\langle W,1\rangle),$$

such that

$$W=(w_1,\ldots,w_m),\ S=(d_1,d_2,\ldots,d_{m-1},1),$$

for all $w_s \geq 0$, $s = 1, \ldots, m$ and all constants d_s satisfy $d_s \geq \overline{d}_s$, $s = 1, \ldots, m-1$, where $C_1 \geq 0$ and $\overline{d}_s$ are positive constants sufficiently large.

Under the assumptions (A1)-(A3), the next proposition says that the classical solution of the system (3.1) remains in $\sum_{\mathfrak{S},3}$ for all t in $[0, T_{\max})$.

Proposition 4.1. *Suppose that the assumptions (A1)-(A3) are satisfied. Then for any* W_0 *in* $\sum_{\mathfrak{S},3}$ *the classical solution* W *of the system* (3.1) *on* $[0, T_{\max}) \times \Omega$ *remains in* $\sum_{\mathfrak{S},3}$ *for all* t *in* $[0, T_{\max})$.

(A4) The total mass of the components $w_1, \ldots, w_m$ is controlled with time, which is ensured by

$$\begin{cases} \sum\limits_{1 \leq s \leq r} \Psi_s(t, x, W, p) \leq 0, \text{ for all } 1 \leq r \leq m \\ \text{for all } (W, p) \in (\mathbb{R}^+)^m \times \mathbb{R}^{Nm} \text{ and a.e. } (t, x) \in Q_T \end{cases}$$

$$\Psi_s :]0, T[\, \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{mN} \to \mathbb{R} \text{ are measurable} \tag{4.1}$$

$$\Psi_s : \mathbb{R}^m \times \mathbb{R}^{mN} \to \mathbb{R} \text{ are locally Lipschitz continuous} \tag{4.2}$$

namely

$$\begin{aligned} &\sum_{1 \leq s \leq m} \left| \Psi_s\left(t, x, W, p\right) - \Psi_s\left(t, x, \hat{W}, \hat{p}\right) \right| \\ \leq \;& K(r) \left(\sum_{1 \leq s \leq m} |w_s - \hat{w}_s| + \sum_{1 \leq s \leq m} \|p_s - \hat{p}_s\| \right) \end{aligned}$$

for a.e. (t, x) and for all $0 \leq |w_s|, |\hat{w}_s|, \|p_s\|, \|\hat{p}_s\| \leq r$.

$$|\Psi_1\left(t, x, W, \nabla W\right)| \leq C_1\left(|w_1|\right) \left(F_1\left(t, x\right) + \|\nabla w_1\|^2 + \sum_{2 \leq j \leq m} \|\nabla w_j\|^{\alpha_j} \right) \tag{4.3}$$

where $C_1 : [0, +\infty) \to [0, +\infty)$ is nondecreasing, $F_1 \in L^1\left(Q_T\right)$ and $1 \leq \alpha_j < 2$

$$|\Psi_s\left(t, x, W, \nabla W\right)| \leq C_s \left(\sum_{j=1}^{s} |w_j| \right) \left(F_s\left(t, x\right) + \sum_{1 \leq j \leq m} \|\nabla w_j\|^2 \right), \; 2 \leq s \leq m \tag{4.4}$$

where $C_s : [0, +\infty) \to [0, +\infty)$ is nondecreasing, $F_s \in L^1\left(Q_T\right)$ for all $2 \leq s \leq m$.

Let us know that if the nonlinearities Ψ do not dependent on the gradient (system (3.1) is semilinear), the existence of global positive solutions have been obtained by Hollis *et all* [18], Hollis and Morgan [19] and Martin and Pierre [25]. One can see that in all of these works, the triangular structure, namely hypotheses (A4) plays an important role in the study of semilinear systems. Indeed, if (A4) does not hold, Pierre and Schmitt [29] proved blow up in finite time of the solutions to some semilinear reaction-diffusion systems.

Where $\Psi = (\Psi_1, \Psi_2)$ depends on the gradient, Alaa and Mounir [8] solved the problem where the triangular structure is satisfied and the growth of Ψ_1 and Ψ_2 with respect to $|\nabla w_1|, |\nabla w_2|$ is sub-quadratic. There exists $1 \leq p < 2$, $C : [0,\infty)^2 \to [0,\infty)$ nondecreasing such that

$$|\Psi_1| + |\Psi_2| \leq C(|w_1|, |w_2|)(1 + |\nabla w_1|^p + |\nabla w_2|^p)$$

About the critical growth with respect to the gradient ($p = 2$), we recall that for the case of a single equation ($d_1 = d_2$ and $\Psi_1 = \Psi_2$), existence results have been proved for the elliptic case in [11, 12]. The corresponding parabolic equations have also been studied by many authors; see for instance [6, 13, 15, 24].

5. Statement of the result

First, we have to clarify in which sense we want to solved problem (3.1).
The existence of global unique solutions for the system (3.1) is to equivalence to existence a w_s, $s = \overline{1,m}$, true for the following theorem:

Theorem 5.1. *Suppose that the hypotheses (A1)-(A4) and (4.1)-(4.4) are satisfied, so it exists unique w_s, $s = \overline{1,m}$ solution of:*

$$\begin{cases} w_s \in C\left([0,T]; L^1(\Omega)\right) \cap L^1\left(0,T; W_0^{1,1}(\Omega)\right), \\ \Psi_s \in L^1(Q_T) \text{ where } Q_T = (0,T)\times\Omega \text{ for all } T > 0, \\ w_s(t) = S_s(t)\, w_s^0 + \int_0^t S_s(t-\tau)\,\Psi_s(s,.,W(\tau),\nabla W(\tau))\,d\tau, \\ s = \overline{1,m},\ \forall t \in [0,T[, \end{cases} \tag{5.1}$$

where $W = (w_1,\ldots,w_m)$, $\nabla W = (\nabla w_1,\ldots,\nabla w_m)$ and $S_s(t)$ are the semigroups of contractions in $L^1(\Omega)$ generated by $\lambda_s\Delta$, $s = \overline{1,m}$.

Example 5.2. For $1 \leq i \leq m$, A typical example where the result of this paper can be applied is

$$\begin{cases} \dfrac{\partial w_i}{\partial t} - d_i\Delta w_i = \sum\limits_{1\leq j\leq i} a_{ij}\dfrac{w_j}{\sum\limits_{1\leq k\leq m} w_k}|\nabla w_j|^2 + f_i(t,x) & \text{in } Q_T \\ w_i = 0 & \text{on } \Sigma_T \\ w_i(0,x) = w_{i,0} & \text{in } \Omega \end{cases}$$

Theorem 5.3. *Assume that (A2), (A4) and (4.1)-(4.4) hold. If $w_s^0 \in L^2(\Omega)$, for all $1 \leq s \leq m$, then there exists a positive global solution $W = (w_1,\ldots,w_m)$ of system (3.1). Moreover, $w_1,\ldots,w_m \in L^2\left(0,T; H_0^1(\Omega)\right)$.*

Before giving the proof of this theorem, let us define the following functions.
Given a real positive number k, we set

$$T_k(s) = \max\{-k, \min(k,s)\} \quad \text{and } G_k(s) = s - T_k(s)$$

We remark that

$$\begin{cases} T_k(s) = s & \text{for } 0 \leq s \leq k, \\ T_k(s) = k & \text{for } s > k. \end{cases}$$

Proof of Theorem 5.3.
Approximating scheme. For every function h defined from $\mathbb{R}^+ \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{mN}$ into $\mathbb{R}$, we associate $\hat{\varphi} = \hat{\varphi}(t, x, W, p)$ such that

$$\hat{\varphi} = \begin{cases} \varphi\left(t, x, w_1, \ldots, w_m, p_1, \ldots, p_m\right) & \text{if } w_s \geq 0,\ 1 \leq s \leq m \\ \begin{array}{l} \varphi\left(t, x, w_1, \ldots, w_{s-1}, 0, w_{s+1}, \ldots, w_m,\right. \\ \left. p_1, \ldots, p_{s-1}, 0, p_{s+1}, \ldots, p_m\right) \end{array} & \text{if } w_s \leq 0 \text{ and } w_j \geq 0,\ j \neq s \\ \varphi\left(t, x, 0, \ldots, 0, p_1, \ldots, p_m\right) & \text{if } w_s \leq 0,\ 1 \leq s \leq m \end{cases}$$

and consider the system, for $1 \leq s \leq m$

$$\begin{cases} \frac{\partial w_s}{\partial t} - d_s \Delta w_s = \hat{\Psi}_s\left(t, x, W, \nabla W\right) & \text{in }]0, +\infty[\times \Omega, \\ w_s = 0 \quad \text{or} \quad \frac{\partial w_s}{\partial \eta} = 0 & \text{on }]0, +\infty[\times \partial\Omega, \\ w_s\left(0, x\right) = w_s^0\left(x\right) & \text{in } \Omega. \end{cases} \tag{5.2}$$

It is obviously seen, by the structure of $\hat{\Psi}_s$, $1 \leq s \leq m$, that systems (3.1) and (5.2) are equivalent on the set where $w_s \geq 0$, $1 \leq s \leq m$. Consequently, to prove Theorem 5.3, we have to show that problem (5.2) has a weak solution which is positive.
To this end, we define ψ_n a truncation function by $\psi_n \in C_c^\infty(\mathbb{R}), 0 \leq \psi_n \leq 1$, and

$$\psi_n(z) = \begin{cases} 1 & \text{if } |z| \leq n \\ 0 & \text{if } |z| \geq n+1 \end{cases}$$

and the mollification with respect to (t, x) is defined as follows.
Let $\rho \in C_c^\infty\left(\mathbb{R} \times \mathbb{R}^N\right)$ such that

$$supp\rho \subset B(0,1), \quad \int \rho = 1, \ \rho \geq 0 \text{ on } \mathbb{R} \times \mathbb{R}^N$$

and $\rho_n(y) = n^N \rho(ny)$. One can see that

$$\rho_n \in C_c^\infty\left(\mathbb{R} \times \mathbb{R}^N\right), \ supp\rho_n \subset B\left(0, \frac{1}{n}\right), \ \int \rho_n = 1, \ \rho_n \geq 0 \text{ on } \mathbb{R} \times \mathbb{R}^N$$

We also consider nondecreasing sequences $w_{s,0}^n \in C_c^\infty\left(\Omega\right)$ such that

$$w_{s,0}^n \to w_s^0 \text{ in } L^2\left(\Omega\right), \ 1 \leq s \leq m$$

and define for all (t, x, W, p) in $\mathbb{R}^+ \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{mN}$ and $1 \leq s \leq m$,

$$\Psi_{s,n}(t, x, W, p) = \left[\psi_n\left(\sum_{1 \leq j \leq m}\left(|w_j| + \|p_j\|\right)\right)\Psi_s(t, x, W, p)\right] * \rho_n\left(t, x\right).$$

Note that these functions enjoy the same properties as Ψ_s, $1 \leq s \leq m$, moreover they are Hölder continuous with respect to (t, x) and $|\Psi_{s,n}| \leq M_n$, $1 \leq s \leq m$, where M_n is a constant depending only on n (these estimates can be derived from (5.1), the properties of the convolution product, and the fact that $\int \rho_n = 1$).
Let us now consider the truncated system, for $1 \leq s \leq m$

$$\begin{cases} \dfrac{\partial w_{s,n}}{\partial t} - d_s \Delta w_{s,n} = \Psi_{s,n}\left(t, x, W_n, \nabla W_n\right) & \text{in } Q_T \\ w_{s,n} = 0 \quad \text{or} \quad \frac{\partial w_{s,n}}{\partial \eta} = 0 & \text{on } \Sigma_T \\ w_{s,n}\left(0, x\right) = w_{s,0}^n\left(x\right) & \text{in } \Omega \end{cases} \tag{5.3}$$

It is well known that problem (5.3) has a global classical solution (see [23], theorem 7.1, p. 591) for the existence and ([24], Corollary of Theorem 4.9, p. 341) for the regularity of solutions. It remains to show the positivity of the solutions.

Lemma 5.4. *Let $w_n = (w_{1,n}, \ldots, w_{m,n})$ be a classical solution of (5.3) and suppose that $w_{1,0}^n, \ldots, w_{m,0}^n \geq 0$. Then $w_{1,n}, \ldots, w_{m,n} \geq 0$.*

Proof. See [8], Lemma 1, p 537. □

5.1. A priori estimates

The hypotheses (A2) and (A4) allowed the following lemma.

Lemma 5.5. *(i) There exists a constant M depending on $\sum_{1\leq j\leq m} \|w_{j,0}\|_{L^1(\Omega)}$ such that*

$$\int_\Omega \left(\sum_{1\leq j\leq m} w_{j,n}(t) \right) \leq M, \text{ for all } t \in [0,T]$$

(ii) There exists a constant R_1 depending on $\sum_{1\leq j\leq m} \|w_{j,0}\|_{L^1(\Omega)}$, such that

$$\sum_{1\leq j\leq m} \int_\Omega |\Psi_{j,n}(t,x,W_n,\nabla W_n)| \leq R_1.$$

(iii) There exists a constant R_2 depending on k and $\sum_{1\leq s\leq m} \left\|w_s^0\right\|_{L^1(\Omega)}$, such that for all $1 \leq j \leq m$

$$\int_{Q_T} |\nabla T_k(w_{j,n})|^2 \leq R_2.$$

(iv) There exists a constant R_3 depending on $\sum_{1\leq j\leq r} \|w_{j,0}\|_{L^2(\Omega)}$ such that for all $2 \leq r \leq m$,

$$\int_{Q_T} \left| \nabla T_k \left(\sum_{1\leq j\leq r} w_{j,n} \right) \right|^2 \leq R_3$$

(v) There exists a constant R_4 depending on $\sum_{1\leq j\leq m} \|w_{j,0}\|_{L^2(\Omega)}$ and $d_1, \ldots, d_m$ such that

$$\int_{Q_T} |\Psi_{j,n}(t,x,W_n,\nabla W_n)| \left(\sum_{1\leq r\leq m} (m-r+1) w_{k,n} \right) \leq R_4, \text{ for all } 1 \leq j \leq m.$$

Proof. See Bouarifi *et al.* [14]. □

5.2. Convergence

Our objective is to show that $W_n = (w_{1,n}, \ldots, w_{m,n})$ converges to some $W = (w_1, \ldots, w_m)$ solution of the problem (5.1). The sequences $w_{1,0}^n, \ldots, w_{m,0}^n$ are uniformly bounded in $L^1(\Omega)$ (since they converge in $L^2(\Omega)$), and by Lemma 5.5, the nonlinearities $\Psi_{1,n}, \ldots, \Psi_{m,n}$ are uniformly bounded in $L^1(Q_T)$. Then according to a result in [10] the applications

$$\left(w_{s,0}^n, \Psi_{s,n}\right) \to w_{s,n},\ 1 \leq s \leq m$$

are compact from $L^1(\Omega) \times L^1(Q_T)$ into $L^1\left(0, T; W_0^{1,1}(\Omega)\right)$.
Therefore, we can extract a subsequence, still denoted by $(w_{1,n}, \ldots, w_{m,n})$, such that

$$\begin{array}{ll} (w_{1,n}, \ldots, w_{m,n}) \to (w_1, \ldots, w_m) & \text{in } L^1\left(0, T; W_0^{1,1}(\Omega)\right) \\ (w_{1,n}, \ldots, w_{m,n}) \to (w_1, \ldots, w_m) & \text{a.e. in } Q_T \\ (\nabla w_{1,n}, \ldots, \nabla w_{m,n}) \to (\nabla w_1, \ldots, \nabla w_m) & \text{a.e. in } Q_T \end{array}$$

Since $\Psi_{1,n}, \ldots, \Psi_{m,n}$ are continuous, we have

$$\Psi_{s,n}(t, x, W_n, \nabla W_n) \to \Psi_s(t, x, W, \nabla W) \qquad \text{a.e. in } Q_T,\ 1 \leq s \leq m.$$

This is not sufficient to ensure that $(w_1, \ldots, w_m)$ is a solution of (5.1). In fact, we have to prove that the previous convergence are in $L^1(Q_T)$. In view of the Vitali theorem, to show that $\Psi_{s,n}(t, x, W_n, \nabla W_n)$, $1 \leq s \leq m$, converges to $\Psi_s(t, x, W, \nabla W)$ in $L^1(Q_T)$, is equivalent to proving that $\Psi_{s,n}(t, x, W_n, \nabla W_n)$, $1 \leq s \leq m$ are equi-integrable in $L^1(Q_T)$

Lemma 5.6. *$\Psi_{s,n}(t, x, W_n, \nabla W_n)$, for all $1 \leq s \leq m$, are equi-integrable in $L^1(Q_T)$.*

The proof of this lemma requires the following result based on some properties of two time-regularization denoted by w_γ and $w_\sigma (\gamma, \sigma > 0)$ which we define for a function $w \in L^2\left(0, T; H_0^1(\Omega)\right)$ such that $w(0) = w_0 \in L^2(\Omega)$ (for more details see [8]). In the following we will denote by $\omega(\varepsilon)$ a quantity that tends to zero as ε tends to zero, and $\omega^\sigma(\varepsilon)$ a quantity that tends to zero for every fixed σ as ε tends to zero.

Lemma 5.7. *Let (w_n) be a sequence in $L^2\left(0, T; H_0^1(\Omega)\right) \cap C([0, T])$ such that $w_n(0) = w_0^n \in L^2(\Omega)$ and $(w_n)_t = \rho_{1,n} + \rho_{2,n}$ with $\rho_{1,n} \in L^2\left(0, T; H^{-1}(\Omega)\right)$ and $\rho_{2,n} \in L^1(Q_T)$. Moreover assume that w_n converges to w in $L^2(Q_T)$, and w_0^n converges to $w(0)$ in $L^2(\Omega)$.*
Let Υ be a function in $C^1([0, T])$ such that $\Upsilon \geq 0$, $\Upsilon' \leq 0, \Upsilon(T) = 0$. Let φ be a Lipschitz increasing function in $C^0(\mathbb{R})$ such that $\varphi(0) = 0$. Then for all $k, \gamma > 0$

$$\begin{aligned} & \left\langle \rho_{1n}, \Upsilon\varphi\left(T_k(w_n) - T_k(w_m)_\gamma\right)\right\rangle + \int_{Q_T} \rho_{2n} \Upsilon\varphi\left(T_k(w_n) - T_k(w_m)_\gamma\right) \\ \geq\ & \omega^{\gamma,n}\left(\frac{1}{m}\right) + \omega^\gamma\left(\frac{1}{n}\right) + \int_\Omega \Upsilon(0)\, \Phi\left(T_k(w) - T_k(w)_\gamma\right)(0)\, dx \\ & - \int_\Omega G_k(w)(0)\, \Upsilon(0)\, \varphi\left(T_k(w) - T_k(w)_\gamma\right)(0)\, dx \end{aligned}$$

where $\Phi(t) = \int_0^t \varphi(s) ds$ and $G_k(s) = s - T_k(s)$

Proof. See [8], Lemma 7, p 544. □

Lemma 5.8. *Suppose that $w_{j,n}, w_j$, $1 \leq j \leq m$, are as above.*

(i) If

$$|\Psi_{1,n}| \leq C_1(|w_{1,n}|)\left(F_1(t,x) + |\nabla w_{1,n}|^2 + \sum_{2\leq j\leq m} |\nabla w_j|^{\alpha_j}\right)$$

where $C_1 : [0,+\infty) \to [0,+\infty)$ is nondecreasing, $F_1 \in L^1(Q_T)$ and $1 \leq \alpha_j < 2$. Then for each fixed k

$$\lim_{n\to\infty} \int_{Q_T} |\nabla T_k(w_{1,n}) - \nabla T_k(w_1)|^2 \chi_{\left[\sum\limits_{1\leq j\leq m} w_{j,n}\leq k\right]} = 0.$$

(ii) If

$$|\Psi_{s,n}(t,x,W,\nabla W)| \leq C_s\left(\sum_{j=1}^{s} |w_j|\right)\left(F_s(t,x) + \sum_{1\leq j\leq m} |\nabla w_j|^2\right),\ 2 \leq s \leq m$$

where $C_s : [0,+\infty) \to [0,+\infty)$ is nondecreasing, $F_s \in L^1(Q_T)$ for all $2 \leq s \leq m$. Then for each fixed k and for all $2 \leq s \leq m$

$$\lim_{n\to\infty} \int_{Q_T} \left|\nabla T_k\left(\sum_{1\leq j\leq s} w_{j,n}\right) - \nabla T_k\left(\sum_{1\leq j\leq s} w_j\right)\right|^2 \chi_{\left[\sum\limits_{1\leq j\leq m} w_{j,n}\leq k\right]} = 0.$$

Proof. (i) This is a direct consequence of the resulting output established in [8, 14] □

Proof of Lemma 5.6. Let A be a measurable subset of Ω, we have

$$\begin{aligned}\int_A |\Psi_{1,n}(t,x,W_n,\nabla W_n)| &= \int_{A\cap[E_n>k]} |\Psi_{1,n}| + \int_{A\cap[E_n\leq k]} |\Psi_{1,n}| \\ &\leq \int_{A\cap[\theta_n>k]} |\Psi_{1,n}| + \int_{A\cap[E_n\leq k]} |\Psi_{1,n}|\end{aligned}$$

with $E_n = \sum\limits_{1\leq j\leq m} w_{j,n}$ and $\theta_n = \sum\limits_{1\leq k\leq m} (m-k+1)w_{k,n}$.

Thanks to (iii) (Lemma 5.5), we obtain $\forall\varepsilon > 0$, $\exists k_0$ such that if $k \geq k_0$ then for all n

$$\begin{aligned}&\int_{A\cap[E_n>k]} |\Psi_{1,n}(t,x,W_n,\nabla W_n)| \\ \leq\ & \frac{1}{k}\int_{[E_n>k]} k|\Psi_{1,n}| \leq \frac{1}{k}\int_{Q_T} E_n|\Psi_{1,n}| \leq \frac{1}{k}\int_{Q_T} \theta_n|\Psi_{1,n}| \leq \frac{\varepsilon}{m+2}\end{aligned}$$

Hypothesis (4.3) implies that for all $k > k_0$

$$\begin{aligned}
&\int_A |\Psi_{1,n}(t,x,W_n,\nabla W_n)| \\
\leq\ & \frac{\varepsilon}{m+2} + C_1(k)\left(\int_A F_1(t,x) + \int_{A\cap[E_n\leq k]} |\nabla w_{1,n}|^2\right) \\
&+C_1(k)\sum_{2\leq j\leq m}\left(\int_{A\cap[E_n\leq k]} |\nabla w_{j,n}|^{\alpha_j}\right) \\
\leq\ & \frac{\varepsilon}{m+2} + C_1(k)\left(\int_A F_1(t,x) + \int_{A\cap[E_n\leq k]} |\nabla T_k(w_{1,n})|^2\right) \\
&\quad +C_1(k)\sum_{2\leq j\leq m}\left(\int_{A\cap[E_n\leq k]} |\nabla T_k(w_{j,n})|^{\alpha_j}\right)
\end{aligned}$$

Using Hölder's inequality for $1 \leq \alpha_j < 2$ and (iii) (Lemma 5.5), we obtain

$$\begin{aligned}
C_1(k)\int_{A\cap[E_n\leq k]} |\nabla T_k(w_{j,n})|^{\alpha_j} &\leq C_1(k)\left(\int_A |\nabla T_k(w_{j,n})|^2\right)^{\frac{\alpha_j}{2}} |A|^{\frac{2-\alpha_j}{2}} \\
&\leq C_1(k)R_2^{\frac{\alpha_j}{2}}|A|^{\frac{2-\alpha_j}{2}} \leq \frac{\varepsilon}{m+2}
\end{aligned}$$

Whenever $|A| \leq \varrho_j$, with $\varrho_j = \left(\frac{\varepsilon}{m+2}C_1^{-1}(k)R_2^{-\frac{\alpha_j}{2}}\right)^{\frac{2}{2-\alpha_j}}$, $2 \leq j \leq m$ To deal with the second integral we write

$$\int_{A\cap[E_n\leq k]} |\nabla T_k(w_{1,n})|^2 \leq 2\int_{A\cap[E_n\leq k]} |\nabla T_k(w_{1,n}) - \nabla T_k(w_1)|^2 + 2\int_A |\nabla T_k(w_1)|^2$$

According to (iii) (Lemma 5.5), $|\nabla T_k(w_{1,n}) - \nabla T_k(w_1)|^2 \chi_{[E_n\leq k]}$ is equi-integrable in $L^1(\Omega)$ since it converges strongly to 0 in $L^1(\Omega)$. So, there exists ϱ_{m+1} such that if $|A| \leq \varrho_{m+1}$, then

$$2C_1(k)\int_{A\cap[E_n\leq k]} |\nabla T_k(w_{1,n}) - \nabla T_k(w_1)|^2 \leq \frac{\varepsilon}{m+2}$$

On the other hand $F_1, |\nabla T_k(w_1)|^2 \in L^1(\Omega)$, therefore there exists ϱ_{m+2} such that

$$C_1(k)\left(2\int_A |\nabla T_k(w_1)|^2 + \int_A F_1(t,x)\right) \leq \frac{\varepsilon}{m+2}$$

whenever $|A| \leq \varrho_{m+2}$. Choose $\varrho_0 = \inf\{\varrho_j,\ 2 \leq j \leq m+2\}$, If $|A| \leq \varrho_0$ we obtain

$$\int_A |\Psi_{1,n}(x,W_n,\nabla W_n)| \leq \varepsilon.$$

Similarly, we get for all $2 \leq s \leq m$

$$\int_A |\Psi_{s,n}| \leq \frac{\varepsilon}{m+2}$$

$$+C_s(k)\left(\int_A F_s(t,x)+\int_{A\cap[E_n\leq k]}\left(6\left|\nabla w_1\right|^2+6\left|\nabla T_k(w_{1,n})-\nabla T_k(w_1)\right|^2\right)\right)$$

$$+8C_s(k)\sum_{2\leq r\leq m}\left(\sum_{A\cap[E_n\leq k]}\left|\nabla T_k\left(\sum_{1\leq j\leq r}w_j\right)\right|^2\right)$$

$$+8C_s(k)\sum_{2\leq r\leq m}\left(\sum_{A\cap[E_n\leq k]}\left|\nabla T_k\left(\sum_{1\leq j\leq r}w_{j,n}\right)-\nabla T_k\left(\sum_{1\leq j\leq r}w_j\right)\right|^2\right)$$

Arguing in the same way as before, we obtain the required result. □

Then $(w_1,\ldots,w_m)$ verify (3.1) consequently $(w_1,\ldots,w_m)$ is the solution of (1.1).

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General decay rates of the solution energy in a viscoelastic wave equation with boundary feedback and a nonlinear source

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Abstract. In a bounded domain, we consider a viscoelastic equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = |u|^\gamma u$$

with a nonlinear feedback localized on a part of the boundary, where $\gamma > 0$ and the relaxation function g satisfied $g'(t) \leq \xi(t)g^p(t),\ 1 \leq p < \frac{3}{2}$, and certain initial data. We establish an explicit and general decay rate result, using some properties of the convex functions. Our new results substantially improve several earlier related results in the literature.

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Keywords: General decay, nonlinear source, viscoelastic, wave equation, relaxation function.

1. Introduction

In this paper, we are concerned with the energy decay rate of the following viscoelastic problem with nonlinear boundary dissipation and a nonlinear source

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = |u|^\gamma u, & in\ \ \Omega \times (0,\infty) \\ u = 0, & on\ \Gamma_0 \times (0,\infty) \\ \frac{\partial u}{\partial \nu} - \int_0^t g(t-\tau)\frac{\partial u}{\partial \nu}(\tau)d\tau + h(u_t) = 0, & on\ \Gamma_1 \times (0,\infty) \\ u(x,0) = u_0(x);\ u_t(x,0) = u_1(x), & x \in \Omega \end{cases} \tag{1.1}$$

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where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint, with meas$(\Gamma_0) > 0$, ν is the unit outward normal to $\partial\Omega$, $\gamma > 0$, and g, h are specific functions.

Let us mention some known results related to the viscoelastic problem with nonlinear boundary dissipation. In [7], Cavalcanti and al. considered the following problem

$$\begin{cases} u_{tt} - \triangle u + \int_0^t g(t-s) \triangle u(s)\, ds = 0, & \text{in } \Omega \times (0,\infty) \\ \frac{\partial u}{\partial \nu} - \int_0^t g(t-s) \frac{\partial u}{\partial \nu}(s)\, ds + h(u_t) = 0, & \text{on } \Gamma_1 \times (0,\infty) \\ u(x,t) = 0, & \text{on } \Gamma_0 \times (0,\infty) \\ u(x,0) = u_0, \;\; u_t(x,0) = u_1, & x \in \Omega. \end{cases} \tag{1.2}$$

The existence and uniform decay rate results were established under quite restrictive assumptions on damping term h and the kernel function g. Later, Cavalcanti and al. [6] generalized this result without imposing a growth condition on h and under a weaker assumption on g. Recently, Messaoudi and Mustafa [18] exploited some properties of convex functions [2] and the multiplier method to extend these results. They established an explicit and general decay rate result without imposing any restrictive growth assumption on the damping term h and greatly weakened the assumption on g. Also, Li et al [11] have analyzed the global existence and decay estimates for nonlinear viscoelastic wave equation with boundary dissipation. They established uniform decay rate of the energy under suitable conditions on the initial data and the relaxation function g. Let us also mention other papers in connection with viscoelastic effects such as Dafermos [8] [9], Mustafa MI [22], Lagnese [10], Aassila et al. [1]. On considering the boundary dissipation,we refer the reader to related works Mohammad M. Al-Gharabli [3], [20], [21], [23] and the references therein.

In a situation in which a source term is competing with the viscoelastic dissipation, many authors have established stability results. For example, Messaoudi [16] looked at

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau) d\tau = |u|^\gamma u, & in \;\; \Omega \times (0,\infty) \\ u = 0, & on\; \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x); \;\; u_t(x,0) = u_1(x), & x \in \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary, $\gamma > 0$, and the relaxation function g is a positive and uniformly decaying function satisfies a relation of the form

$$g'(t) \leq -\xi(t) g(t), \tag{1.3}$$

where ξ is a nonincreasing differentiable function such that

$$\left|\frac{\xi'(t)}{\xi(t)}\right| \leq k, \quad \xi(t) > 0, \;\; \xi'(t) \leq 0, \;\; \forall t > 0, \qquad \int_0^\infty \xi(t) dt = +\infty.$$

He established a more general decay result, from which the usual exponential and polynomial decay rates are only special cases.

In a situation in which a source term is competing with the viscoelastic dissipation and on considering the boundary dissipation, Shun and Hsueh [24] considered

$$u_{tt} - k_0 \triangle u(t) + \int_0^t g(t-s)\, div\,(a(x)\nabla u(s))\, ds + b(x)\, u_t = f(u),\ \text{ in } \Omega \times (0,\infty),$$
$$k_0 \frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\,(a(x)\nabla u(s)).\nu ds + h(u_t) = 0, \text{on } \Gamma_1 \times (0,\infty),$$
$$u(x,t) = 0,\ \text{ on } \Gamma_0 \times (0,\infty),$$
$$u(x,0) = u_0,\quad u_t(x,0) = u_1,\ x \in \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary, the relaxation function g is a positive and uniformly decaying function satisfying (1.3), and where ξ is a nonincreasing differentiable positive function such that

$$\int_0^\infty \xi(t)dt = +\infty.$$

The authors established the general decay rate of the solution energy which is not necessarily of exponential or polynomial type. Another problems, in which in which a source term is competing with the viscoelastic dissipation and on considering the boundary dissipation, were discussed in [5], [14], [12] and [13], and the existence, uniform decay rate results were established.

In this article, we devote ourselves to the study of the problem (1.1). Motivated by previous work and by the idea of Messaoudi and Mustafa [17], which considers a wider class of relaxation functions g, we obtain a more general and explicit energy decay formula, to from which the exponential and the polynomial decay rates are only special cases of our result. In fact, our decay formulas extend and improve some results of the literature.

2. Preliminaries

In this section we prepare some material needed in the proof of our result. We have the imbedding: $H^1_{\Gamma_0} \hookrightarrow L^{2(\gamma+1)}(\Omega)$. Let $C_e > 0$ be the optimal constant of Sobolev imbedding which satisfies the following inequality:

$$\|u\|_{2(\gamma+1)} \leq C_e \|\nabla u\|_2,\quad \forall u \in H^1_{\Gamma_0}, \tag{2.1}$$

and we use the trace-Sobolev imbedding: $H^1_{\Gamma_0} \hookrightarrow L^k(\Gamma_1)$, $1 \leq k < \frac{2(n-1)}{n-2}$. In this case, the imbedding constant is denoted by B_1, that is

$$\|u\|_{k,\Gamma_1} \leq B_1 \|\nabla u\|_2. \tag{2.2}$$

Next, we state the assumptions for problem (1.1) as follows.

For the relaxation function g we assume the following:
(G_1) $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a nonincreasing C^1 function satisfying

$$g(0) > 0,\qquad 1 - \int_0^\infty g(s)ds = l > 0.$$

(G_2) There exists a nonincreasing differentiable function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\xi(0) > 0$, and satisfying

$$g'(t) \leq \xi(t) g^p(t), \quad 1 \leq p < \frac{3}{2}, \quad t \geq 0.$$

(G_3) For the nonlinear term, we assume

$$0 < \gamma \leq \frac{2}{(n-2)}, \quad n \geq 3$$
$$\gamma > 0, \quad n = 1, 2.$$

(G_4) $h : \mathbb{R} \longrightarrow \mathbb{R}$ is a nondecreasing C^0 function such that there exist a strictly increasing function $h_0 \in C^1([0, +\infty))$, with $h_0(0) = 0$, and positive constants c_1, c_2, ϵ such that

$$h_0(|\, s \,|) \leq |\, h(s) \,| \leq h_0^{-1}(|\, s \,|) \quad for\ all \quad |\, s \,| \leq \epsilon$$
$$c_1 \,|\, s \,| \leq |\, h(s) \,| \leq c_2 \,|\, s \,| \qquad for\ all \quad |\, s \,| \geq \epsilon.$$

In addition, we assume that the function H, defined by $H(s) = \sqrt{s} h_0(\sqrt{s})$, is a strictly convex C^2 function on $(0, r^2]$, for some $r > 0$, when h_0 is nonlinear.

By using the Galerkin method and procedure similar to that of [11], and [23], we have the following local existence result for problem (1.1).

Theorem 2.1. *Let hypotheses* (G1)-(G4) *hold and assume that* $u_0 \in H^1_{\Gamma_0} \cap H^2(\Omega)$, $u_1 \in H^1_{\Gamma_0}$. *Then there exists a strong solution* u *of* (1.1) *satisfying*

$$\begin{aligned} u &\in L^\infty\left([0, T) ; H^1_{\Gamma_0} \cap H^2(\Omega)\right) \\ u_t &\in L^\infty\left([0, T) ; H^1_{\Gamma_0}\right) \\ u_{tt} &\in L^\infty\left([0, T) ; L^2(\Omega)\right), \end{aligned}$$

for some $T > 0$.

Proposition 2.2. *Suppose that* (G1), (G3) *and* (G4) *hold. Let* $(u_0, u_1) \in V \times L^2(\Omega)$ *be given, satisfying* (2.7). *Then the solution* u *of* (1.1) *is global and bounded.*

We introduce the following functionals

$$\begin{aligned} J(t) &= \frac{1}{2}\left(k_1 - \int_0^t g(s)\, ds\right) \|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{\gamma + 2} \|u\|_{\gamma+2}^{\gamma+2} \\ E(t) &= J(u(t)) + \frac{1}{2} \|u_t\|_2^2, \quad \text{for } t \in [0, T) \qquad (2.3) \\ I(t) &= I(u(t)) = \left(k_1 - \int_0^t g(s)\, ds\right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t) - \|u\|_{\gamma+2}^{\gamma+2}, \quad (2.4) \end{aligned}$$

where

$$(g \circ v)(t) = \int_0^t g(t-s) \|v(t) - v(s)\|_2^2\, ds, \qquad (2.5)$$

and $E(t)$ is the energy functional.

A direct differentiation, using (1.1), leads to

$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2} g(t) \parallel \nabla u(t) \parallel_2^2 - \int_{\Gamma_1} u_t(t) h(u_t(t)) d\Gamma \leq 0, \qquad (2.6)$$

For completeness, using similar procedure in [15], we state the global existence result.

Lemma 2.3. *Suppose that* $(G1)$ *and* $(G3)$ *hold, and* $(u_0, u_1) \in V \times L^2(\Omega)$*, such that*

$$\beta = \frac{C_e^{\gamma+2}}{l}\left(\frac{2(\gamma+2)}{\gamma l}E(u_0,u_1)\right)^{\gamma/2} < 1 \tag{2.7}$$

$$I(u_0) > 0,$$

then $I(u(t)) > 0$, $\forall t > 0$.

Proposition 2.4. *Suppose that* $(G1)$, $(G3)$ *and* $(G4)$ *hold. Let* $(u_0, u_1) \in V \times L^2(\Omega)$ *be given, satisfying* (2.7)*. Then the solution* u *of* (1.1) *is global and bounded.*

Adopting the proof of [17], we have the following results which are crucial for the proof of our main result.

Lemma 2.5. *Assume that* g *satisfies* $(G1)$ *and* $(G2)$ *then*

$$\int_0^{+\infty} \xi(t)\, g^{1-\sigma}(t)\, dt < +\infty, \quad \forall \sigma < 2 - p.$$

Lemma 2.6. *Assume that* g *satisfies* $(G1)$ *and* $(G2)$*, and* u *is the solution of* (1.1) *then, for* $0 < \delta < 1$*, we have*

$$(g \circ \nabla u)(t) \le C\left[\left(\int_0^{+\infty} g^{1-\sigma}(t)\, dt\right) E(0)\right]^{\frac{p-1}{p-1+\delta}} (g^p \circ \nabla u)^{\frac{\delta}{p-1+\delta}}(t).$$

By taking $\delta = \frac{1}{2}$*, we get*

$$(g \circ \nabla u)(t) \le C\left[\int_0^t g^{\frac{1}{2}}(s)\, ds\right]^{\frac{2p-2}{2p-1}} (g^p \circ \nabla u)^{\frac{1}{2p-1}}(t). \tag{2.8}$$

Corollary 2.7. *Assume that* g *satisfies* $(G1)$ *and* $(G2)$*, and* u *is the solution of* (1.1) *then*

$$\xi(t)(g \circ \nabla u)(t) \le C\left[-E'(t)\right]^{\frac{1}{2p-1}}. \tag{2.9}$$

If G is a convex function on $[a, b]$, ($-G$ is convex), $f : \Omega \to [a, b]$ and h are integrable functions on Ω, with $h(x) \ge 0$ and $\int_\Omega h(x)\, dx = k > 0$, then Jensen's inequality states that

$$\frac{1}{k}\int_\Omega G[f(x)]\, h(x)\, dx \le G\left[\frac{1}{k}\int_\Omega f(x)\, h(x)\, dx\right]. \tag{2.10}$$

For the special case $G(y) = y^{\frac{1}{q}}$, $y \ge 0$, $p > 1$, we have

$$\frac{1}{k}\int_\Omega [f(x)]^{\frac{1}{q}}\, h(x)\, dx \le \left[\frac{1}{k}\int_\Omega f(x)\, h(x)\, dx\right]^{\frac{1}{q}}.$$

3. Decay of solutions

In this section we state and prove the main result of our work. For this purpose, we adopt the following result from [24] without proof.

Lemma 3.1. *There exist positive constants ε_1, ε_2, m, t_0 such that the fun*

$$F(t) := E(t) + \varepsilon_1 \psi_1(t) + \varepsilon_2 \psi_2(t), \tag{3.1}$$

is equivalent to E and satisfies

$$F'(t) \leq -mE(t) + c \int_{\Gamma_1} h^2(u_t) d\tau + c(g \circ \nabla u)(t). \tag{3.2}$$

where

$$\psi_1(t) := \int_\Omega u u_t dx, \tag{3.3}$$

$$\psi_2(t) := -\int_\Omega u_t \int_0^t g(t-\tau)(u(t) - u(\tau)) d\tau dx.$$

Lemma 3.2. [19] *Under the assumptions $(G1)$, $(G2)$ and $(G4)$, the solution satisfies the estimates*

$$\int_{\Gamma_1} h^2(u_t) d\Gamma \leq \int_{\Gamma_1} u_t h(u_t) d\Gamma, \qquad \text{if } h_0 \text{ is linear} \tag{3.4}$$

$$\int_{\Gamma_1} h^2(u_t) d\Gamma \leq c H^{-1}(J(t)) - cE'(t), \qquad \text{if } h_0 \text{ is nonlinear} \tag{3.5}$$

where

$$J(t) = \frac{1}{|\Gamma_{12}|} \int_{\Gamma_{12}} u_t h(u_t) d\Gamma \leq E'(t),$$

and

$$\Gamma_{12} = \{x \in \Gamma_1 : |u_t| \leq \varepsilon_1\}.$$

Proof. **Case 1:** h_0 is linear, using $(G4)$ we have

$$c_1' |u_t| \leq | h(u_t) | \leq c_2' |u_t|,$$

and hence

$$h^2(u_t) \leq c_2' u_t h(u_t).$$

So, (3.4) is established.

Case 2: h_0 is nonlinear on $[0, \varepsilon]$:

First, we assume that $\max\{r, h_0(r)\} < \varepsilon$; otherwise we take r smaller. Let $\varepsilon_0 = \min\{r, h_0(r)\}$; them for $\varepsilon_0 \leq | s | \leq \varepsilon$, using $(G4)$, we have

$$| h(s) | \leq \frac{h_0^{-1}(| s |)}{| s |} | s | \leq \frac{h_0^{-1}(\varepsilon)}{\varepsilon_0} | s | \quad \text{and} \quad | h(s) | \geq \frac{h_0(| s |)}{| s |} | s | \geq \frac{h_0(\varepsilon_0)}{\varepsilon} | s |,$$

so, we conclude that

$$\begin{cases} h_0(| s |) \leq | h(s) | \leq h_0^{-1}(| s |) & \text{for all } | s | < \varepsilon_0 \\ c_1' | s | \leq | h(s) | \leq c_2' | s | & \text{for all } | s | \geq \varepsilon_0. \end{cases} \tag{3.6}$$

Since $H(s^2) =\mid s \mid h_0(\mid s \mid)$, then using (3.6), we obtain

$$H(h^2(s)) \leq sh(s) \quad for\ all \ \mid s \mid \leq \varepsilon_0,$$

which gives

$$h^2(s) \leq H^{-1}(sh(s)) \quad for\ all \ \mid s \mid \leq \varepsilon_0.$$

To estimate the last integral in (3.2), we consider the following partition of Γ_1:

$$\Gamma_{11} = \{x \in \Gamma_1 : |u_t| > \varepsilon_0\}, \qquad \Gamma_{12} = \{x \in \Gamma_1 : |u_t| \leq \varepsilon_0\}.$$

Recalling the definition of ε_0 and using (3.6), we obtain on Γ_{12},

$$u_t h(u_t) \leq \varepsilon_0 h_0^{-1}(\varepsilon_0) \leq h_0(r)r = H(r^2) \tag{3.7}$$

and

$$u_t h(u_t) \leq \varepsilon_0 h_0^{-1}(\varepsilon_0) \leq r h_0^{-1} h_0(r) = r^2.$$

Jensen's inequality gives

$$H^{-1}(J(t)) \geq c \int_{\Gamma_{12}} H^{-1}(u_t h(u_t)) d\Gamma. \tag{3.8}$$

Thus, using (3.6) – (3.8), we get

$$\begin{aligned}
\int_{\Gamma_1} h^2(u_t) d\Gamma &= \int_{\Gamma_{12}} h^2(u_t) d\Gamma + \int_{\Gamma_{11}} h^2(u_t) d\Gamma \\
&\leq \int_{\Gamma_{12}} H^{-1}(u_t h(u_t)) d\Gamma + c \int_{\Gamma_{11}} u_t h(u_t) d\Gamma \\
&\leq c H^{-1}(J(t)) - c E'(t).
\end{aligned} \tag{3.9}$$

□

Theorem 3.3. *Let* $(u_0, u_1) \in \left(H^1_{\Gamma_0} \times L^2(\Omega)\right)$ *be given . Assume that* $(G1)$*-*$(G4)$ *are satisfied and* h_0**is linear**. *Then, for any* $t_0 > 0$, *there exist two positive constants* K, *and* λ *such that the solution of* (1.1) *satisfies, for all* $t \geq t_0$,

$$E(t) \leq K e^{-\lambda \int_{t_0}^{t} \xi(s) ds}, \qquad if\ p = 1. \tag{3.10}$$

$$E(t) \leq K \left[\frac{1}{1 + \int_{t_0}^{t} \xi^{2p-1}(s)\, ds} \right]^{\frac{1}{2p-2}}, \quad 1 < p < \frac{3}{2}. \tag{3.11}$$

Moreover, if

$$\int_0^{+\infty} \left[\frac{1}{t\xi^{2p-1}(t) + 1} \right]^{\frac{1}{2p-2}} dt < +\infty, \quad 1 < p < \frac{3}{2}, \tag{3.12}$$

then

$$E(t) \leq K \left[\frac{1}{1 + \int_{t_0}^{t} \xi^{p}(s)\, ds} \right]^{\frac{1}{p-1}}, \qquad 1 < p < \frac{3}{2}. \tag{3.13}$$

Proof. Multiplying (3.2) by $\xi(t)$ and using Eqs. 3.4, we get

$$\begin{aligned}\xi(t)F'(t) &\leq -m\xi(t)E(t)+c\xi(t)(g\circ\nabla u)(t)+c\xi(t)\int_{\Gamma_1}h^2(u_t)d\tau\\ &\leq -m\xi(t)E(t)+c\xi(t)(g\circ\nabla u)(t)-c\xi(t)E'(t)\end{aligned}$$

which gives, as $\xi(t)$ is non-increasing,

$$(\xi\mathcal{F}+CE)'(t)\leq -m\xi(t)E(t)+c\xi(t)(g\circ\nabla u)(t),\quad \forall t\geq t_0. \tag{3.14}$$

Let $L(t):=\xi(t)\mathcal{F}(t)+CE(t)$, then clearly $L\sim E$ and we have, for some $m_1>0$,

$$L'(t)\leq -m_1\xi(t)L(t)+c\xi(t)(g\circ\nabla u)(t),\ \forall t\geq t_0.$$

Now, using the procedure similar to that of [17], we obtain the results of the theorem.

□

Theorem 3.4. *Let* $(u_0,u_1)\in V\times L^2(\Omega)$ *be given, satisfying* (2.7). *Assume that* $(G1)-(G4)$ *hold and* h_0 **is nonlinear**. *Then there exist positive constants* k_1, k_2 *and* k_3*such that the solution of* (1.1) *satisfies , for all* $t\geq t_0$,

$$E(t)\leq k_3H_1^{-1}\left(k_1\int_{t_0}^t\xi(s)ds+k_2\right),\qquad p=1. \tag{3.15}$$

$$E(t)\leq k_3H_1^{-1}\left(k_1\int_{t_0}^t\xi^{2p-1}(s)ds+k_2\right),\quad 1<p<\frac{3}{2}. \tag{3.16}$$

Moreover, if

$$\int_0^{+\infty}H_1^{-1}\left(k_1t\xi^{2p-1}(t)+k_2\right)dt<+\infty,\quad 1<p<\frac{3}{2}, \tag{3.17}$$

then

$$E(t)\leq k_3H_2^{-1}\left(k_1\int_{t_0}^t\xi^p(s)ds+k_2\right),\qquad 1<p<\frac{3}{2}, \tag{3.18}$$

where $H_1(t)=\int_t^1\dfrac{1}{t^{2p-1}H'(\varepsilon_0t))}ds.$ and where $H_2(t)=\int_t^1\dfrac{1}{t^{2p-1}H'(\varepsilon_0t))}ds.$
Here, H_1 and H_2 are strictly decreasing and convex on $(0,1]$, with $\lim\limits_{t\longrightarrow0}H_i(t)=+\infty$, $i=1,2$.

Simple calculations show that (3.16) and (3.17) yield

$$\int_{t_0}^{+\infty}E(t)\,dt<+\infty.$$

Proof. **Case of** $p=1$. Recalling $G(2)$ and (2.6), Multiplying (3.2) by $\xi(t)$, we obtain, for all $t\geq t_0$

$$\begin{aligned}\xi(t)\mathcal{F}'(t) &\leq -m\xi(t)E(t)+C(\xi(t)g\circ\nabla u)(t)+c\xi(t)\int_{\Gamma_1}h^2(u_t)d\tau \qquad (3.19)\\ &\leq -m\xi(t)E(t)-C(g'\circ\nabla u)(t)+c\xi(t)\int_{\Gamma_1}h^2(u_t)d\tau\\ &\leq -m\xi(t)E(t)-CE'(t)+c\xi(t)\int_{\Gamma_1}h^2(u_t)d\tau,\end{aligned}$$

which leads to

$$(\xi\mathcal{F}+CE)'(t) \leq -m\xi(t)E(t) + c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau, \quad \forall t \geq t_0. \tag{3.20}$$

Let $L(t) := \xi(t)\mathcal{F}(t) + CE(t)$, then clearly $L \sim E$ and we have, for some $m_1 > 0$,

$$L'(t) \leq -m_1\xi(t)L(t) + c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau, \ \forall t \geq t_0.$$

Now, using the procedure similar to that of [19], we obtain the results of the theorem.

Case of $1 < p < \dfrac{3}{2}$.

Multiplying (3.2) by $\xi(t)$ and we using 2.7, we obtain

$$\xi(t)F'(t) \leq -m\xi(t)E(t) + c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau + k(-E'^{\frac{1}{2p-1}}(t)),$$

multiplying by $\xi^{2p-2}(t)E^{2p-2}(t)$ and using Young's inequality

$$\begin{aligned}\xi^{2p-1}(t)E^{2p-2}(t)F'(t) &\leq -m\xi^{2p-1}(t)E^{2p-1}(t) + c\xi^{2p-1}(t)E^{2p-2}(t)\int_{\Gamma_1} h^2(u_t)d\tau \\ &+ k\left(-E'(t)\right)^{\frac{1}{2p-1}}(t)\xi^{2p-2}(t)E^{2p-2}(t) \\ &\leq -m\xi^{2p-1}(t)E^{2p-1}(t) + c\xi^{2p-1}(t)E^{2p-2}(t)\int_{\Gamma_1} h^2(u_t)d\tau \\ &+ k(-E'^{2p-1}(t)E^{2p-1}(t) \\ F_2'(t) &\leq k_1\xi^{2p-1}(t)E^{2p-1}(t) + c\xi^{2p-1}(t)E^{2p-2}(t)\int_{\Gamma_1} h^2(u_t)d\tau.\end{aligned} \tag{3.21}$$

With $F_2(t) = F(t)\xi^{2p-1}(t)E^{2p-2}(t) + kE(t); \quad F_0 \sim E$.
Therefore, using $(3.5), (2.5)$ becomes

$$F_2'(t) \leq k_1\xi^{2p-1}(t)E^{2p-1}(t) + c\xi^{2p-1}(t)E^{2p-2}(t)\left(H^{-1}(\lambda(t)) - E'(t)\right)$$
$$F_2'(t) \leq k_1\xi^{2p-1}(t)E^{2p-1}(t) + c\xi^{2p-1}(t)E^{2p-2}(t)H^{-1}(\lambda(t)) - c\xi^{2p-1}(0)E^{2p-2}(0)E'(t)$$
$$F_3'(t) \leq k_1\xi^{2p-1}(t)E^{2p-1}(t) + c\xi^{2p-1}(t)E^{2p-2}(t)H^{-1}(\lambda(t))$$

with $F_3 = F_2 + CE$ then, $F_3 \sim E$.
Now, for $\epsilon_0 < r^2$ and $c_0 > 0$, using (3.9) and the fact that $E' \leq 0, \ H' \geq 0, \ H'' \geq 0$ on $(0, r^2]$, we find that the functional F_2 defined by

$$F_4(t) := H'\left(\epsilon_0\frac{E(t)}{E(0)}\right)F_2(t) + c_0E(t)$$

satisfies, for some $\alpha_1, \alpha_2 > 0$,

$$\alpha_1F_4(t) \leq E(t) \leq \alpha_2F_4(t) \tag{3.22}$$

and

$$\begin{aligned}
F_4'(t) &= \epsilon_0 \frac{E'(t)}{E(0)} H''\left(\epsilon_0 \frac{E(t)}{E(0)}\right) F_2(t) + H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) F_2'(t) + c_0 E'(t) \\
&\leq -k\xi^p E^p(t) H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + c\xi^p(t) E^{p-1}(t) H^{-1}(\lambda(t)) H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + c_0 E'(t).
\end{aligned} \tag{3.23}$$

Let H^* be the convex conjugate of H in the sense of young (see [4] p. $61 - 64$); then

$$H^*(s) = s\left(H'\right)^{-1}(s) - H\left[\left(H'\right)^{-1}(s)\right], \quad if\ s \in \left[0, H'^2)\right]$$

and H^* satisfies the following Young's inequality:

$$AB \leq H^*(A) + H(B), \quad if\ A \in \left(0, H'^2)\right],\ B \in \left(0, r^2\right]. \tag{3.24}$$

With $A = H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)$ and $B = H^{-1}(\lambda(t))$, using (2.6), (3.7) and (3.23) – (3.24), we arrive at

$$\begin{aligned}
F_4'(t) \quad \leq \quad & -k\xi^{2p-1} E^{2p-1}(t) H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + c\xi(t)\lambda(t) \\
& + c\xi^{2p-1}(t) E^{2p-2}(t) H^*\left(H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)\right) + c_0 E'(t),
\end{aligned}$$

that gives

$$\begin{aligned}
F_4'(t) \quad \leq \quad & -k\xi^{2p-1} E^{2p-1}(t) H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + c\epsilon_0 \xi^{2p-1}(t) \frac{E^{2p-1}(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \\
& - cE'(t) + c_0 E'(t).
\end{aligned}$$

Consequently, with a suitable choice of ϵ_0 and k, we obtain, for all $t \geq t_0$,

$$F_4'(t) \leq -k_1 \xi^{2p-1}(t) \left(\frac{E(t)}{E(0)}\right)^{2p-1} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) = -k_1 \xi^{2p-1}(t) H_2\left(\frac{E(t)}{E(0)}\right), \tag{3.25}$$

where $H_2(t) = t^{2p-1} H'(\epsilon_0 t)$.
Since $h_0 \in C^1([0, +\infty])$, then it is evident that $H \in C^1([0, +\infty])$ and $H'(0) = h_0'(0)$. So, $H_2(0) = 0$ and since

$$H_2'(t) = (2p-1) t^{2p-2} H'(\epsilon_0 t) + \epsilon_0 t^{2p-1} H''(\epsilon_0 t)$$

then, using the strict convexity of H on $(0, r^2]$, we find that $H_2'(t), H_2(t) > 0$ on $[0, 1]$. Thus, with $R(t) = \frac{\alpha_1 F_4(t)}{E(0)}$, and using (3.22) and (3.25), we have $R \sim E$ and, for some $k_1 > 0$,

$$R'(t) \leq -k_1 \xi^{2p-1}(t) H_2(R(t)), \quad \forall\ t \geq t_0.$$

Then, a simple integration gives, for some $k_2 > 0$,

$$R(t) \leq H_1^{-1}\left(k_1 \int_{t_0}^{t} \xi^{2p-1}(s) ds + k_2\right), \quad \forall t > t_0,$$

where $H_1(t) = \int_t^1 \dfrac{1}{H_2(s)} ds$.
To establish (3.18) Multiplying(3.2) by $\xi(t)$ and recall Remark 3. So, we have

$$\xi(t)\mathcal{F}'(t) \leq -m\xi(t)E(t) + C\xi(t)(g \circ \nabla u)(t) + c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau$$
$$= -m\xi(t)E(t) + C\frac{\eta(t)}{\eta(t)}\int_0^t [\xi^p(s)g^p(s)]^{\frac{1}{p}} \|\nabla u(t) - \nabla u(t-s)\|_2^2$$
$$+ c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau, \tag{3.26}$$

where

$$\begin{aligned}
\eta(t) &= \int_0^t \|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \leq C\int_0^t \|\nabla u(t)\|_2^2 + \|\nabla u(t-s)\|_2^2 ds \\
&\leq C\int_0^t [E(t) + E(t-s)] ds \leq 2C\int_0^t E(t-s) ds \\
&= 2C\int_0^t E(s) ds < 2C\int_0^{+\infty} E(s) ds < +\infty.
\end{aligned}$$

Applying Jensens's inequality (2.10) for the second term on the right hand side of (3.26), with

$$G(y) = y^{\frac{1}{p}},\ y > 0,\ f(s) = \xi^p(s)g^p(s)$$

and

$$h(s) = \|\nabla u(t) - \nabla u(t-s)\|_2^2,$$

to get

$$\begin{aligned}
\xi(t)F'(t) &\leq -m\xi(t)E(t) + c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau, \\
&\quad + C\eta(t)\left[\frac{1}{\eta(t)}\int_0^t \xi^p(s)g^p(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds\right]^{\frac{1}{p}}
\end{aligned}$$

where we assume that $\eta(t) > 0$.
Therefore, we obtain

$$\begin{aligned}
\xi(t)F'(t) &\leq -m\xi(t)E(t) + c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau \\
&\quad + C\eta^{\frac{p-1}{p}}(t)\left[\xi^{p-1}(0)\int_0^t \xi(s)g^p(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds\right]^{\frac{1}{p}} \\
&\leq -m\xi(t)E(t) + C(-g' \circ \nabla u)^{\frac{1}{p}}(t) + c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau \\
&\leq -m\xi(t)E(t) + C(-E'(t))^{\frac{1}{p}} + c\xi(t)\int_{\Gamma_1} h^2(u_t)d\tau.
\end{aligned}$$

Multiplying by $\xi^p(t)\,E^p(t)$, and repeating the same computations as in above, we arrive at

$$E(t) \le k_3 H_2^{-1}\left(k_1 \int_{t_0}^{t} \xi^p(s)ds + k_2\right), \qquad 1 < p < \frac{3}{2},$$

where $H_2(t) = \int_t^1 \dfrac{1}{t^p H'(\varepsilon_0 t))} ds.$

Remark 3.5. In the case where $\| \nabla u(t) - \nabla u(t-s) \| = 0$ and hence from (3.2) we have

$$\mathcal{F}'(t) \le -mE(t) + c\int_{\Gamma_1} h^2(u_t)d\tau,$$

using the procedure similar to that of [19], we obtain

Case h_0 linear

$$E(t) \le Ce^{-mt}.$$

Case h_0 nonlinear

$$E(t) \le H_1^{-1}(k_1 t + k_2), \quad \forall t > t_0$$

This completes the proof of our main result. □

Example 3.6. As in [17], we give an example to illustrate the existence of relaxation function g and ξ satisfying $(G2)$:

If $p = 1$**:**

Let $g(t) = ae^{-b(1+t)}$, where $b > 0 < \nu \le 1$ and $a > 0$ is chosen so that $\int_0^{+\infty} g(t)\,dt < 1$. Then $g'(t) = -\xi(t)\,g(t)$ where $\xi(t) = b$.

If $1 < p < \frac{3}{2}$**:**

Let $g(t) = \frac{a}{(1+t)^\nu}$, $\nu > 2$, where $a > 0$ is a constant so that $\int_0^{+\infty} g(t)\,dt < 1$. We have

$$g'(t) = -\frac{a\nu}{(1+t)^{\nu+1}} = -b\left(\frac{a}{(1+t)^\nu}\right)^{\frac{\nu+1}{\nu}} = -bg^p(t), \quad p = \frac{\nu+1}{\nu} < \frac{3}{2}, \quad b > 0.$$

with $\xi(t) = b$.

Example 3.7. As in [2, 6], we give an example to illustrate the energy decay rates given by **Theorem** (3.3) and **Theorem** (3.4).
If h satisfies

$$c_1 \min\{| s |, | s |^q\} \le | h(s) | \le c_2 \max\left\{| s |, | s |^{1/q}\right\},$$

for some c_1, $c_2 > 0$ and $q \ge 1$. Then $h_0(s) = cs^q$ and $\bar{H}(s) = \sqrt{s}h_0(\sqrt{s}) = cs^{\frac{q+1}{2}}$ is a strictly convex C^2 function on $(0, \infty)$, then $H_1^{-1}(t) = (ct + c_1)^{\frac{-2}{4p+q-5}}$, and the relaxation function g and ξ given in Example 3.6.

Then, we obtain for some constants $c,\ c',\ c''>0$:
If $p=1$ and $q=1$ **(**h_0 **is linear)**, by **Theorem** (3.3) we arrive at

$$E(t)\le ce^{-c'\int_0^t \xi(s)ds}=ce^{-c'bt}.$$

If $1<p<\frac{3}{2}$ and $q=1$ **(**h_0 **is linear)**, by **Theorem** (3.3) we arrive at

$$E(t)\le c\left(c'\int_0^t \xi^{2p-1}(s)ds+c''\right)^{-\frac{1}{2p-2}}=c\left(c'bt+c''\right)^{-\frac{1}{2p-2}}.$$

If $p=1$ and $q>1$ **(**h_0 **is nonlinear)**, by **Theorem** (3.4) we arrive at

$$E(t)\le c\left(c'\int_0^t \xi(s)ds+c''\right)^{-\frac{2}{q-1}}=c\left(c'bt+c''\right)^{-\frac{2}{q-1}}.$$

If $1<p<\frac{3}{2}$ and $q>1$ **(**h_0 **is nonlinear)**, by **Theorem** (3.4) we arrive at

$$E(t)\le c\left(c'\int_0^t \xi^{2p-1}(s)ds+c''\right)^{-\frac{2}{4p+q-5}}=c\left(c'bt+c''\right)^{-\frac{2}{4p+q-5}}.$$

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Asymptotic behavior of generalized $CR-$iteration algorithm and application to common zeros of accretive operators

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Abstract. The purpose of this study is to provide a generalized $CR-$iteration algorithm for finding common fixed points (CFP_s) for nonself quasi-nonexpansive mappings ($QNEMs$) in a uniformly convex Banach space. The suggested algorithm's convergence analysis is analyzed in uniformly convex Banach spaces.

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1. Introduction

Let $\mathfrak{B}$ be a Banach space, $\emptyset \neq \mathfrak{B}_s \subseteq \mathfrak{B}$ be closed and convex, and $\Upsilon : \mathfrak{B}_s \to \mathfrak{B}_s$ be an operator which has at least one fixed point. Then, for the initial value $\mathfrak{a}_0 \in \mathfrak{B}_s$:

(i) Picard's iteration algorithm [16] is defined as :
$\mathfrak{a}_{\eta+1} = \Upsilon \mathfrak{a}_\eta, \ \forall \ \eta \in \mathbb{N}_0.$

(ii) Mann's iteration algorithm [13] is defined as:
$\mathfrak{a}_{\eta+1} = (1-\kappa_\eta)\mathfrak{a}_\eta + \kappa_\eta \Upsilon \mathfrak{a}_\eta, \ \forall \ \eta \in \mathbb{N}_0,$
where $\{\kappa_\eta\} \in (0,1)$.

(iii) Ishikawa's iteration algorithm [8] is defined as:
$\mathfrak{a}_{\eta+1} = (1-\kappa^1_\eta)\mathfrak{a}_\eta + \kappa^1_\eta \Upsilon[(1-\kappa^2_\eta)\mathfrak{a}_\eta + \kappa^2_\eta \Upsilon \mathfrak{a}_\eta], \ \forall \ \eta \in \mathbb{N}_0,$
where $\{\kappa^1_\eta\}$ and $\{\kappa^2_\eta\} \in (0,1)$.

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For nonexpansive operators, it is very well established that the Picard iteration algorithm often does not work effectively. As a result, for the estimation of FPs for nonexpansive type mappings in ambient spaces, the Mann and Ishikawa iterative algorithms have been extensively studied (see [1, 3, 6]).
On the other side, Chug et al. [5] introduced the $CR-$iteration algorithm in a Banach space in 2012. The structure of the $CR-$iterative algorithm differs significantly from that of the Mann and Ishikawa iterative algorithms, making it absolutely independent of both. Several mathematicians have been intrigued by the $CR-$iterative algorithm as an alternative iterative algorithm for fixed point analysis in recent years (see [9, 2]), and it has opened up a substantial field of research in various aspects (see [11, 12]).

Let Υ be a self map on $\mathfrak{B}$. Then the sequence $\{\mathfrak{a}_\eta\}_{n=0}^{\infty}$ defined as follows:

$$\begin{cases} \mathfrak{a}_0 \in \mathfrak{B} \\ \mathfrak{a}_{\eta+1} = (1-\kappa_\eta^1)\mathfrak{b}_\eta + \kappa_\eta^1 \Upsilon \mathfrak{b}_\eta, \\ \mathfrak{b}_\eta = (1-\kappa_\eta^2)\Upsilon \mathfrak{a}_\eta + \kappa_\eta^2 \Upsilon \mathfrak{c}_\eta, \\ \mathfrak{c}_\eta = (1-\kappa_\eta^3)\mathfrak{a}_\eta + \kappa_\eta^3 \Upsilon \mathfrak{a}_\eta, \end{cases} \tag{CR}$$

where $\{\kappa_\eta^1\}$, $\{\kappa_\eta^2\}$ and $\{\kappa_\eta^3\} \in (0,1)$ is called $CR-$iteration. The $CR-$iteration method is a three-step iteration method. For contraction mappings, $CR-$iterative algorithms perform better than Picard and Ishikawa iterative algorithms, and behave well for nonexpansive mappings.
We are concerned with two quasi-nonexpansive nonself mappings $\mathcal{M}_1,\ \mathcal{M}_2 : \mathfrak{B}_s \to \mathfrak{B}$, where $\mathfrak{B}_s$ is a nonempty subset of the Banach space $\mathfrak{B}$, the iterative location and weak limits of the proposed iterative algorithm for these types of functions in the context of current research [19]. Our findings are applied to the zeros of accretive operators in some different ways.

2. Tools and notations

In this section, we discuss the notations which we are going to use in the entire manuscript. The framework in which we shall prove our results from now on is a Banach space $\mathfrak{B}$. Υ is a mapping. $\mathbb{N}_0$ represents the set of natural numbers including 0, whereas the terminology $\mathbb{R}$ is used to represent the set of real numbers. The notation 'for all'is represented by '$\forall$'and 'such that'is represented by '$\ni$'. The symbol $\in$ represents 'belongs to'. The terminology $\mathcal{H}_s$ is used to represent the 'Hilbert space'with the inner product $\langle\cdot,\cdot\rangle$ and whereas $\mathcal{Q}_{\mathfrak{B}_s}$ is a retraction of $\mathfrak{B}$ onto $\mathfrak{B}_s$. $\mathcal{P}_{\mathfrak{B}_s}$ is used to represent the projection from $\mathfrak{B}$ to $\mathfrak{B}_s$. $\mathcal{H}_s^{'} \subseteq \mathcal{H}_s$. $Dom(\mathcal{A})$ represents the domain of $\mathcal{A}$, $Ran(\mathcal{A})$ is used to represent the range set of $\mathcal{A}$, and $Gr(\mathcal{A})$ is the graph of $\mathcal{A}$ whereas $\mathcal{A}^{-1}$ is the inverse of $\mathcal{A}$. Δ is a non-negative real number. The terminology 'fixed points', we denote by 'FP_s'.The Proximal point algorithm is denoted by 'PPA'. It is important to note that the 'set of all fixed points'is denoted by '$\digamma_{(\Upsilon)}$'. Furthermore, ∇ is used to represent the 'vector differential operator'.

3. Preliminaries

In this section, we discuss key definitions and lemmas that are necessary in order to make this article self-contained.
Throughout the paper, we denote the closed ball with the center at $\mathfrak{a}$ and radius r by $\mathcal{CB}_r[\mathfrak{a}]$ and is defined as

$$\mathcal{CB}_r[\mathfrak{a}] = \{\mathfrak{b} \in \mathfrak{B} : ||\mathfrak{a} - \mathfrak{b}|| \leq r\}.$$

Also, $\mathfrak{B}$ is said to be uniformly convex if for $0 < \epsilon \leq 2$, $||\mathfrak{a}|| \leq 1$, $||\mathfrak{b}|| \leq 1$ and $||\mathfrak{a} - \mathfrak{b}|| \geq \epsilon$ imply $\exists\ \mu = \mu(\epsilon) > 0 \ni$

$$\frac{1}{2}||\mathfrak{a} + \mathfrak{b}|| \leq 1 - \mu.$$

Lemma 3.1. [21] *Let $\mathfrak{m} > 1$ and $r_1 > 0$ be two fixed numbers. Then, $\mathfrak{B}_s$ is uniformly convex iff $\exists$ a convex and strictly increasing function $\Upsilon : [0, \infty) \to [0, \infty)$ with $\Upsilon(0) = 0 \ni$*

$$||\mathfrak{c}\mathfrak{a} + (1 - \mathfrak{c})\mathfrak{b}||^{\mathfrak{m}} \leq \mathfrak{c}||\mathfrak{a}||^{\mathfrak{m}} + (1 - \mathfrak{c})||\mathfrak{b}||^{\mathfrak{m}} - \mathfrak{c}(1 - \mathfrak{c})\Upsilon(||\mathfrak{a} - \mathfrak{b}||),$$

$\forall\ \mathfrak{a}, \mathfrak{b} \in \mathfrak{B}_{\mathfrak{m}} > [0]$ *and* $\mathfrak{c} \in [0, 1]$.

For $\mathcal{H}_s$, we have

$$||\mathfrak{c}\mathfrak{a} + (1 - \mathfrak{c})\mathfrak{b}||^2 \leq \mathfrak{c}||x||^2 + (1 - \mathfrak{c})||y||^2 - \mathfrak{c}(1 - \mathfrak{c})||\mathfrak{a} - \mathfrak{b}||,$$

$\forall\ \mathfrak{a}, \mathfrak{b} \in \mathcal{H}_s$ and $\mathfrak{c} \in [0, 1]$.

Definition 3.2. A mapping $\Upsilon : \mathfrak{B}_s \to \mathfrak{B}$ has the demiclosed property at $\mathfrak{b} \in \mathfrak{B}$ if

$$\{\mathfrak{a} \in \mathfrak{B}_s, \mathfrak{a}_\eta \to \mathfrak{a} \text{ and } \Upsilon\mathfrak{a}_\eta \to \mathfrak{b} \implies \mathfrak{a} \in \mathfrak{B}_s \text{ and } \Upsilon\mathfrak{a} = \mathfrak{b}\}.$$

Lemma 3.3. [4]*Let $\mathfrak{B}_s$ be a nonempty,closed and convex subset of a uniformly convex Banach space $\mathfrak{B}$.If $\Upsilon : \mathfrak{B}_s \to \mathfrak{B}$ is nonexpansive mappings then $I - \Upsilon$ has the demiclosed property with respect to* 0.

The collection of points of $\mathfrak{B}_s$, unaltered by Υ is defined as follows:

$$F_{(\Upsilon)} = \{\mathfrak{a} \in \mathfrak{B}_s : \Upsilon\mathfrak{a} = \mathfrak{a}\}.$$

For a constant $L \in [0, \infty)$, the mapping Υ is called $L-$Lipschitz if

$$||\Upsilon\mathfrak{a} - \Upsilon\mathfrak{b}|| \leq L||\mathfrak{a} - \mathfrak{B}||,$$

$\forall\ \mathfrak{a}, \mathfrak{b} \in \mathfrak{B}_s$. Every $1-$Lipschitz is called $QNEM$.

A retract of $\mathfrak{B}$ is a subset $\mathfrak{B}_s$ of a Banach space $\mathfrak{B}$ that has a continuous mapping $\mathcal{Q}_{\mathfrak{B}_s}$ from $\mathfrak{B}$ to $\mathfrak{B}_s$ such that $\mathcal{Q}_{\mathfrak{B}_s}(\mathfrak{a}) = \mathfrak{a}$ for any $\mathfrak{a} \in \mathfrak{B}_s$. A $\mathcal{Q}_{\mathfrak{B}_s}$ like this is known as $\mathfrak{B}$ onto $\mathfrak{B}_s$ retraction.
If $\mathcal{Q}_{\mathfrak{B}_s}(\mathcal{Q}_{\mathfrak{B}_s}(\mathfrak{a} + \mathfrak{c}(\mathfrak{a} - \mathcal{Q}_{\mathfrak{B}_s}(\mathfrak{a})))) = \mathcal{Q}_{\mathfrak{B}_s}(\mathfrak{a})$, $\forall\ \mathfrak{a} \in \mathfrak{B}$ and $\mathfrak{c} \geq 0$, a retraction $\mathcal{Q}_{\mathfrak{B}_s}$ is said to be sunny. $\mathfrak{B}_{\mathfrak{s}}$ is a sunny nonexpansive retract of $\mathfrak{B}$ if a sunny retraction $\mathcal{Q}_{\mathfrak{B}_s}$ is also nonexpansive. Let $\mathfrak{B}$ be reflexive and strictly convex Banach space. Let $\mathcal{P}_{\mathfrak{B}_{\mathfrak{s}}} : \mathfrak{B} \to \mathfrak{B}_{\mathfrak{s}}$ be a projection. Also, $\mathcal{P}_{\mathfrak{B}_{\mathfrak{s}}}(\mathfrak{a})$ is in $\mathfrak{B}_s$ with the property

$$||\mathfrak{a} - \mathcal{P}_{\mathfrak{B}_{\mathfrak{s}}}(\mathfrak{a})|| = \{\inf ||\mathfrak{a} - \mathfrak{u}|| : \mathfrak{u} \in \mathfrak{B}_s\}.$$

for $\mathfrak{a} \in \mathfrak{B}$.

It is also well comprehended that $\mathcal{P}_{\mathcal{H}'_s}(\mathfrak{a}) \in \mathcal{H}_s$ and

$$\langle \mathfrak{a} - \mathcal{P}_{\mathcal{H}'_s}(\mathfrak{a}),\ \mathcal{P}_{\mathcal{H}'_s}(\mathfrak{a}) - \mathfrak{b}\rangle \geq 0,$$

$\forall\ \mathfrak{a} \in \mathcal{H}_s$, $\mathfrak{b} \in \mathcal{H}'_s$.
Sunny nonexpansive retractions work in the same way in $\mathfrak{B}$ as projections do in $\mathcal{H}_s$. If a subset $\mathcal{H}'_s \neq \emptyset$ of $\mathcal{H}$ is closed and convex, then $\exists$ a unique sunny nonexpansive retraction from $\mathfrak{B}_s$ to $\mathcal{H}'_s$.

Definition 3.4. [1]Let $\mathfrak{B}$ be a Banach space. For any sequence $\{\mathfrak{a}_\eta\} \to \mathfrak{a} \in \mathfrak{B}$, and $\forall$ $\mathfrak{b} \neq \mathfrak{a}$, we say that $\mathfrak{B}$ satisfies the Opial condition, if the following inequality holds:

$$\lim sup_{\eta\to\infty}||\mathfrak{a}_\eta - \mathfrak{a}|| < \lim sup_{\eta\to\infty}||\mathfrak{a}_\eta - \mathfrak{b}||.$$

It is to be noted that $\lim sup$ can be substituted by $\lim inf$ in this definition and that every Hilbert space satisfies the Opial condition [1]. Let $\emptyset \neq \mathfrak{B}_s \subseteq \mathfrak{B}$, $\Upsilon : \mathfrak{B}_s \to \mathfrak{B}$ a mapping, and $\{\mathfrak{a}_\eta\}$ a sequence in $\mathfrak{B}_s$. If $\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \Upsilon\mathfrak{a}_\eta|| = 0$, then $\{\mathfrak{a}_\eta\}$ is referred to as a sequence in Υ.
The following proposition is the generalization of Proposition 2.5 [20].

Proposition 3.5. *Let $\Upsilon : \mathfrak{B}_s \to \mathfrak{B}$ be uniformly continuous mapping and $\{\mathfrak{a}_\eta\} \subset \mathfrak{B}_s$ be a sequence of Υ. Then, $\{\mathfrak{b}_\eta\} \subset \mathfrak{B}_s$ is an approximating FP sequence of Υ whenever $\{\mathfrak{b}_\eta\} \in \mathfrak{B}_s \ni \lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \mathfrak{b}_\eta|| = 0$.*

For dual space $\mathfrak{B}^*$ of $\mathfrak{B}$, the symbol $||\cdot||$ denotes the norms of $\mathfrak{B}$ and $\mathfrak{B}^*$. For $\mathfrak{a}^* \in \mathfrak{B}^*$ and $\mathfrak{a} \in \mathfrak{B}$, we use $\langle \mathfrak{a},\ \mathfrak{a}^*\rangle$ instead of $\mathfrak{a}^*(\mathfrak{a})$. The set-valued mapping $J : \mathfrak{B} \to 2^{\mathfrak{B}^*}$ is defined as

$$J(\mathfrak{a}) = \{\mathfrak{a}^* \in \mathfrak{B} : \langle \mathfrak{a},\ \mathfrak{a}^*\rangle = ||\mathfrak{a}||||\mathfrak{a}|| \text{ and } ||\mathfrak{a}^*|| = ||\mathfrak{a}||\}, \qquad \mathfrak{a} \in \mathfrak{B},$$

and is known as a normalized duality mapping of $\mathfrak{B}$. For a multi- valued operator $\mathcal{A} : \mathfrak{B} \to 2^{\mathfrak{B}}$, the following are defined as:

$$Dom(\mathcal{A}) = \{\mathfrak{a} \in \mathfrak{B} : \mathcal{A}\mathfrak{a} \neq \emptyset\},$$
$$Ran(\mathcal{A}) = \cup\{\mathcal{A}\mathfrak{u} : \mathfrak{u} \in Dom(\mathcal{A})\},$$

and

$$Gr(\mathcal{A}) = \{(\mathfrak{a}, \mathfrak{b}) \in \mathfrak{B} \times \mathfrak{B} : \mathfrak{a} \in Dom(\mathcal{A}),\ \mathfrak{b} \in \mathcal{A}\mathfrak{a}\}$$

respectively. $\mathcal{A} \subseteq \mathfrak{B} \times \mathfrak{B}$ represents $\mathcal{A} : \mathfrak{B} \to 2^{\mathfrak{B}}$ and the inverse $\mathcal{A}^{-1}$ of $\mathcal{A}$ is as follows:

$$\mathfrak{a} \in \mathcal{A}^{-1}\mathfrak{b} \iff \mathfrak{b} \in \mathcal{A}\mathfrak{a}.$$

If $\forall\ \mathfrak{a}_i \in Dom(\mathcal{A})$ and $\mathfrak{b} \in \mathcal{A}\mathfrak{a}_i$ for $i = 1, 2$, $\exists\ \jmath \in J(\mathfrak{a}_1 - \mathfrak{a}_2) \ni \langle \mathfrak{b}_1 - \mathfrak{b}_2,\ \jmath\rangle \geq 0$, then the operator is known as *accretive.*

An accretive operator is the negation of a dissipative operator. If there is no proper accretive extension of $\mathcal{A}$, it is known as "maximal accretive", and if $Ran(I + \mathcal{A}) = \mathfrak{B}$, where I symbolizes the identity operator on $\mathfrak{B}$. If $\mathcal{A}$ is "$m-$accretive", then it is maximally accretive. For accretive $\mathcal{A}$, the single-valued nonexpansive mapping $\forall\ \Delta > 0$ is

$$J^{\mathcal{A}}_{\Delta} : Ran(I + \Delta\mathcal{A}) \to Dom(\mathcal{A}), \qquad J^{\mathcal{A}}_{\Delta} = (I + \Delta\mathcal{A})^{-1},$$

and is said to be the *resolvent* of $\mathcal{A}$. The resolvent for an $m-$accretive operator on $\mathfrak{B}$

$$J_{\Delta}^{\mathcal{A}} = (I + \Delta\mathcal{A})^{-1}$$

is a multi-valued nonexpansive mapping whereby the domain is the entire space $\mathfrak{B}$, $\forall\ \Delta > 0$.

Lemma 3.6. [7] *Let $\mathcal{A} : \mathfrak{B} \to 2^{\mathfrak{B}}$ be an $m-$accretive operator. Then $\mathcal{A}$ is the maximal accretive, where $\mathfrak{B}$ is a real Banach space.*

Lemma 3.7. [1] *If $\mathcal{A} : \mathcal{H}_s \to 2^{\mathcal{H}_s}$ is a monotone operator, then $\mathcal{A}$ is the maximal monotone iff $Ran(I + \Delta\mathcal{A}) = \mathcal{H}\ \forall\ \Delta > 0$.*

As a result, if $\mathcal{A} : \mathcal{H}_s \to 2^{\mathcal{H}_s}$ is a maximum monotone operator and $\Delta > 0$, we may define the resolvent of $\mathcal{A}$, $J_{\Delta}^{\mathcal{A}} : \mathcal{H}_s \to: \mathcal{H}_s$, using Lemma 3.7. Also, $J_{\Delta}^{\mathcal{A}}$ satisfies the following inequality

$$||J_{\Delta}^{\mathcal{A}} - \mathfrak{a}J_{\Delta}^{\mathcal{A}}\mathfrak{b}||^2 \leq ||\mathfrak{a} - \mathfrak{b}||^2 - ||(I - J_{\Delta}^{\mathcal{A}})\mathfrak{a} - (I - J_{\Delta}^{\mathcal{A}})\mathfrak{b}||,$$

$\forall\ \mathfrak{a},\ \mathfrak{b} \in \mathcal{H}_s$.

For a function $\wp : \mathcal{H}_s \to (\infty, \infty]$,the domain is defined by:

$$dom(\wp) = \{\mathfrak{a} \in \mathcal{H}_s : \wp(\mathfrak{a}) < \infty\}.$$

Lemma 3.8. [3] *Let $\wp \in \Gamma_0(H)$. Then, $\wp$ is maximal monotone.*

4. Main results

The $CR-$iteration approach allows us to compute the common FP_s of two operators. Our objective is to analyze the asymptotic behaviour of our designed algorithm in Banach spaces. Let $\Upsilon_1,\ \Upsilon_2 : \mathfrak{B} \to \mathfrak{B}_s$ be mappings with at least one common FP between Υ_1 and Υ_2. The collection of common FP_s of mappings Υ_2 and Υ_1 is denoted by $\digamma_{(\Upsilon_2,\Upsilon_1)}$.
We now present the $G - CR-$iteration algorithm, which is as follows:

$$\begin{cases} \mathfrak{a}_0 \in \mathfrak{B}_s, \\ \mathfrak{a}_{\eta+1} = \mathcal{Q}_{\mathfrak{B}_s}[(1 - \kappa_\eta^1)\mathfrak{b}_\eta + \kappa_\eta^1\Upsilon_1\mathfrak{b}_\eta], \\ \mathfrak{b}_\eta = \mathcal{Q}_{\mathfrak{B}_s}[(1 - \kappa_\eta^2)\Upsilon_2\mathfrak{a}_\eta + \kappa_\eta^2\Upsilon_1\mathfrak{c}_\eta], \\ \mathfrak{c}_\eta = \mathcal{Q}_{\mathfrak{B}_s}[(1 - \kappa_\eta^3)\mathfrak{a}_\eta + \kappa_\eta^3\Upsilon_2\mathfrak{a}_\eta], \end{cases} \tag{$G-CR$}$$

where the sequences $\{\kappa_\eta^1\}$, $\{\kappa_\eta^2\}$, $\{\kappa_\eta^3\} \in (0, 1)$. The sequence $\{\mathfrak{a}_\eta\}$ defined by $G-CR$ is called the generalized $CR-$iteration algorithm for mappings Υ_1 and Υ_2. If $\Upsilon_1 = \Upsilon_2$, then $G - CR$ iterative algorithm is defined as follows:

$$\begin{cases} \mathfrak{a}_0 \in \mathfrak{B}_s, \\ \mathfrak{a}_{\eta+1} = \mathcal{Q}_{\mathfrak{B}_s}[(1 - \kappa_\eta^1)\mathfrak{b}_\eta + \kappa_\eta^1\Upsilon_1\mathfrak{b}_\eta], \\ \mathfrak{b}_\eta = \mathcal{Q}_{\mathfrak{B}_s}[(1 - \kappa_\eta^2)\Upsilon_1\mathfrak{a}_\eta + \kappa_\eta^2\Upsilon_1\mathfrak{c}_\eta], \\ \mathfrak{c}_\eta = \mathcal{Q}_{\mathfrak{B}_s}[(1 - \kappa_\eta^3)\mathfrak{a}_\eta + \kappa_\eta^3\Upsilon_1\mathfrak{a}_\eta], \end{cases}$$

where $\{\kappa_\eta^1\}$. $\{\kappa_\eta^2\}$ and $\{\kappa_\eta^3\}$ are sequences in $(0,1)$. To prove the main results, we start with the following lemma.

Lemma 4.1. *Let $\mathcal{Q}_{\mathfrak{B}_s}$ be the sunny nonexpansive retraction and $\Upsilon_1,\ \Upsilon_2 : \mathfrak{B}_s \to \mathfrak{B}$ be QNEM $\ni F_{(\Upsilon_2,\ \Upsilon_1)} \neq \emptyset$. Let $\{\kappa_\eta^1\}$, $\{\kappa_\eta^2\}$, and $\{\kappa_\eta^3\}$ be sequences of real numbers $\ni$ $0 < \kappa_\eta^1,\ \kappa_\eta^2,\ \kappa_\eta^3 < 1$, $\forall\ \eta \in \mathbb{N} \cup\ \{0\}$. Let the sequence $\{\mathfrak{a}_\eta\}$ be generated from $\mathfrak{a}_0 \in \mathfrak{B}_s$ and be defined by $G-CR$. Then, for each $\sigma \in F_{(\Upsilon_2,\Upsilon_1)}$, $\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \sigma||$ exists and*

$$\begin{aligned} &||\mathfrak{b}_\eta - \sigma|| \leq ||\mathfrak{a}_\eta - \sigma||, \ \textit{and} \\ &||\mathfrak{c}_\eta - \sigma|| \leq ||\mathfrak{a}_\eta - \sigma||, \quad \forall \eta \in \mathbb{N} \cup\ \{0\}. \end{aligned} \tag{4.1}$$

Proof. Let σ be a common FP of Υ_1 and Υ_2. Then, for $\eta \in \mathbb{N} \cup\ \{0\}$, the following inequalities hold:

$$\begin{aligned} ||\mathfrak{a}_{\eta+1} - \sigma|| &= ||\mathcal{Q}_{\mathfrak{B}_s}[(1-\kappa_\eta^1)\mathfrak{b}_\eta + \kappa_\eta^1 \Upsilon_1 \mathfrak{b}_\eta] - \mathcal{Q}_{\mathfrak{B}_s}[\sigma]|| \\ &\leq ||(1-\kappa_\eta^1)(\mathfrak{b}_\eta - \sigma) + \kappa_\eta^1(\Upsilon_1\mathfrak{b}_\eta - \sigma)|| \\ &\leq (1-\kappa_\eta^1)||\mathfrak{b}_\eta - \sigma|| + \kappa_\eta^1||\Upsilon_1\mathfrak{b}_\eta - \sigma|| \\ &\leq (1-\kappa_\eta^1)||\mathfrak{b}_\eta - \sigma|| + \kappa_\eta^1||\mathfrak{b}_\eta - \sigma|| \\ &= ||\mathfrak{b}_\eta - \sigma||. \end{aligned} \tag{4.2}$$

Also,

$$\begin{aligned} ||\mathfrak{b}_\eta - \sigma|| &= ||\mathcal{Q}_{\mathfrak{B}_s}[(1-\kappa_\eta^2)\Upsilon_2\mathfrak{a}_\eta + \kappa_\eta^2 \Upsilon_1 \mathfrak{c}_\eta] - \mathcal{Q}_{\mathfrak{B}_s}[\sigma]|| \\ &\leq ||(1-\kappa_\eta^2)(\Upsilon_2\mathfrak{a}_\eta - \sigma) + \kappa_\eta^2(\Upsilon_1\mathfrak{c}_\eta - \sigma)|| \\ &\leq ||(1-\kappa_\eta^2)(\mathfrak{a}_\eta - \sigma) + \kappa_\eta^2(\mathfrak{c}_\eta - \sigma)|| \\ &\leq (1-\kappa_\eta^2)||\mathfrak{a}_\eta - \sigma|| + \kappa_\eta^2||\mathfrak{c}_\eta - \sigma||. \end{aligned} \tag{4.3}$$

Similarly,

$$\begin{aligned} ||\mathfrak{c}_\eta - \sigma|| &= ||\mathcal{Q}_{\mathfrak{B}_s}[(1-\kappa_\eta^3)\mathfrak{a}_\eta + \kappa_\eta^3 \Upsilon_2 \mathfrak{a}_\eta] - \mathcal{Q}_{\mathfrak{B}_s}[\sigma]|| \\ &\leq ||(1-\kappa_\eta^3)(\mathfrak{a}_\eta - \sigma) + \kappa_\eta^3(\Upsilon_2\mathfrak{a}_\eta - \sigma)|| \\ &\leq (1-\kappa_\eta^3)||\mathfrak{a}_\eta - \sigma|| + \kappa_\eta^3||\Upsilon_2\mathfrak{a}_\eta - \sigma|| \\ &\leq (1-\kappa_\eta^3)||\mathfrak{a}_\eta - \sigma|| + \kappa_\eta^3||\mathfrak{a}_\eta - \sigma|| \\ &= ||\mathfrak{a}_\eta - \sigma||. \end{aligned} \tag{4.4}$$

Using inequality (4.4) in (4.3), we have

$$||\mathfrak{b}_\eta - \sigma|| \leq ||\mathfrak{a}_\eta - \sigma||. \tag{4.5}$$

Hence, the inequality (4.2) results

$$||\mathfrak{a}_{\eta+1} - \sigma|| \leq ||\mathfrak{a}_\eta - \sigma||. \tag{4.6}$$

Considering (4.6) and (4.2), we calculate the following result

$$||\mathfrak{a}_{\eta+1} - \sigma|| \leq ||\mathfrak{a}_\eta - \sigma|| \leq ||\mathfrak{a}_{\eta-1} - \sigma|| \leq \ \ldots\ \leq ||\mathfrak{a}_0 - \sigma||, \tag{4.7}$$

$\forall\ \eta \in \mathbb{N} \cup\ \{0\}$. Since $\{||\mathfrak{a}_\eta - \sigma||\}$ is monotonically decreasing, it confirms the convergence of $\{||\mathfrak{a}_\eta - \sigma||\}$. $\square$

The convergence behaviour for $QNEMs$ is now studied by the following theorem.

Theorem 4.2. *Let* $\emptyset \neq \mathfrak{B}_s \subseteq \mathfrak{B}$, *with* $\mathcal{Q}_{\mathfrak{B}_s}$ *as the sunny nonexpansive retraction. Let* $\Upsilon_1,\ \Upsilon_2 : \mathfrak{B}_s \to \mathfrak{B}$ *be* $QNEMs \ni F_{(\Upsilon_1,\Upsilon_2)} \neq \emptyset$. *Let the real sequences* $\{\kappa_\eta^1\}$, $\{\kappa_\eta^2\}$ *and* $\{\kappa_\eta^3\} \ni 0 < a \leq \kappa_\eta^1 \leq \overline{a} < 1$, $0 < b \leq \kappa_\eta^2 \leq \overline{b} < 1$ *and* $0 < c \leq \kappa_\eta^3 \leq \overline{c} < 1$ $\forall$ $\eta \in \mathbb{N} \cup \{0\}$. *Let* $\mathfrak{a}_0 \in \mathfrak{B}_s$ *and* $\mathcal{P}_{F_{(\Upsilon_1,\Upsilon_2)}}(\mathfrak{a}_0) = \mathfrak{a}^*$. *Let* $\{\mathfrak{a}_\eta\}$ *be the sequence defined by* $(G-CR)$. *Then, we have*

1. $\{\mathfrak{a}_\eta\}$ *is in a closed convex bounded set* $\mathcal{CB}_r[\mathfrak{a}^*] \cap \mathfrak{B}_s$, *where* $r \in (0,\infty) \ni ||\mathfrak{a}_0 - \mathfrak{a}^*|| \leq r$.

2. *If* Υ *be uniformly continuous, then*
$$\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \Upsilon_1\mathfrak{a}_\eta|| = 0 \ \textit{and} \ \lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \Upsilon_2\mathfrak{a}_\eta|| = 0,$$
then $\wp_c : [0,\infty) \to [0,\infty)$, $\wp(0) = 0$, *where error bounds are as follows-*
$$\underline{a}(1-\overline{a})\sum_{i=0}^{\eta} \wp_c(||\mathfrak{b}_i - \Upsilon_1\mathfrak{b}_i||) \leq ||\mathfrak{a}_0 - \mathfrak{a}^*||^2 - ||\mathfrak{a}_{\eta+1} - \mathfrak{a}^*||^2, \tag{4.8}$$
$$\underline{b}(1-\overline{b})\sum_{i=0}^{\eta} \wp_c(||\Upsilon_2\mathfrak{a}_i - \Upsilon_1\mathfrak{c}_i||) \leq ||\mathfrak{a}_0 - \mathfrak{a}^*||^2 - ||\mathfrak{a}_{\eta+1} - \mathfrak{a}^*||^2 - \sum_{i=0}^{\eta} \kappa_i^1(1-\kappa_i^1)\wp_c(||\mathfrak{b}_i - \Upsilon_1\mathfrak{b}_i||), \tag{4.9}$$
$$\underline{bc}(1-\overline{c})\sum_{i=0}^{\eta} \wp_c(||\mathfrak{a}_i - \Upsilon_2\mathfrak{a}_i||) \leq ||\mathfrak{a}_0 - \mathfrak{a}^*||^2 - ||\mathfrak{a}_{\eta+1} - \mathfrak{a}^*||^2 - \sum_{i=0}^{\eta} \kappa_i^2(1-\kappa_i^2)\wp_c(||\Upsilon_2\mathfrak{a}_i - \Upsilon_1\mathfrak{c}_i||) - \sum_{i=0}^{\eta} \kappa_i^1(1-\kappa_i^1)\wp_c(||\mathfrak{b}_i - \Upsilon_1\mathfrak{b}_i||), \tag{4.10}$$
$\forall\ \eta \in \mathbb{N} \cup \{0\}$.

3. *If* $I - \Upsilon_2$ *and* $I - \Upsilon_1$ *are demiclosed at* 0 *and* $\mathfrak{B}$ *satisfies the Opial condition, then* $\{\mathfrak{a}_\eta\} \to \ell$ *where* $\ell \in F_{(\Upsilon_2,\Upsilon_1)} \cap \mathcal{CB}_r[\mathfrak{a}^*]$, *where the convergence is weak.*

Proof. (1) Let $\mathfrak{a}^* \in F_{(\Upsilon_2,\Upsilon_1)}$. From inequality (4.7) the following holds for all $\eta \in \mathbb{N} \cup \{0\}$.
$$||\mathfrak{a}_{\eta+1} - \mathfrak{a}^*|| \leq ||\mathfrak{a}_\eta - \mathfrak{a}^*|| \leq ||\mathfrak{a}_{\eta-1} - \mathfrak{a}^*|| \leq \ \ldots \ \leq ||\mathfrak{a}_0 - \mathfrak{a}^*||.$$
Hence, $\{\mathfrak{a}_\eta\} \in \mathcal{CB}_r[\mathfrak{a}^*] \cap \mathfrak{B}_s$.

(2) Let Υ_2 be uniformly continuous. By Lemma 4.1, we have that $\{\mathfrak{a}_\eta\}, \{\mathfrak{b}_\eta\}$ and $\{\mathfrak{c}_\eta\} \in \mathcal{CB}_r[\mathfrak{a}^*] \cap \mathfrak{B}_s$, and hence, from inequality (4.1), we have
$$||\Upsilon_2\mathfrak{a}_\eta - \mathfrak{a}^*|| \leq r, \quad ||\Upsilon_1\mathfrak{a}_\eta - \mathfrak{a}^*|| \leq r, \quad ||\Upsilon_1\mathfrak{b}_\eta - \mathfrak{a}^*|| \leq r \ \text{ and } \ ||\Upsilon_1\mathfrak{c}_\eta - \mathfrak{a}^*|| \leq r,$$

$\forall\ \eta \in \mathbb{N} \cup\ \{0\}$.

Let $\wp_c$ be the function as defined in Lemma 1 for $\mathfrak{m} = 2$ and $r_1 = r$. Benefiting from inequality (4.1) as well, we have

$$\begin{aligned}
||\mathfrak{a}_{\eta+1} - \mathfrak{a}^*||^2 &= ||\mathcal{Q}_{\mathfrak{B}_s}[(1-\kappa_\eta^1)\mathfrak{b}_\eta + \kappa_\eta^1 \Upsilon_1 \mathfrak{b}_\eta] - \mathcal{Q}_{\mathfrak{B}_s}[\mathfrak{a}^*]||^2 \\
&\leq ||(1-\kappa_\eta^1)(\mathfrak{b}_\eta - \mathfrak{a}^*) + \kappa_\eta^1(\Upsilon_1\mathfrak{b}_\eta - \mathfrak{a}^*)||^2 \\
&\leq (1-\kappa_\eta^1)||\mathfrak{b}_\eta - \mathfrak{a}^*||^2 + \kappa_\eta^1||\Upsilon_1\mathfrak{b}_\eta - \mathfrak{a}^*||^2 - \kappa_\eta^1(1-\kappa_\eta^1)\wp_c(||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta||) \\
&\leq (1-\kappa_\eta^1)||\mathfrak{b}_\eta - \mathfrak{a}^*||^2 + \kappa_\eta^1||\mathfrak{b}_\eta - \mathfrak{a}^*||^2 - \kappa_\eta^1(1-\kappa_\eta^1)\wp_c(||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta||) \\
&= ||\mathfrak{b}_\eta - \mathfrak{a}^*||^2 - \kappa_\eta^1(1-\kappa_\eta^1)\wp_c(||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta|| \qquad (4.11) \\
&\leq ||\mathfrak{a}_\eta - \mathfrak{a}^*||^2 - \kappa_\eta^1(1-\kappa_\eta^1)\wp_c(||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta||,
\end{aligned}$$

$\forall\ \eta \in \mathbb{N} \cup\ \{0\}$. By the bounds of sequence $\{\kappa_\eta^1\}$, we have

$$\kappa_\eta^1(1-\kappa_\eta^1)\wp_c(||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta||) \leq ||\mathfrak{a}_\eta - \mathfrak{a}^*||^2 - ||\mathfrak{a}_{\eta+1} - \mathfrak{a}^*||^2.q$$

Observe that

$$\underline{a}(1-\overline{a})\sum_{\eta=0}^{\infty}\wp_c(||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta||) \leq ||\mathfrak{a}_0 - \mathfrak{a}^*|| < \infty.$$

We obtain that $\lim_{\eta\to\infty} ||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta|| = 0$. Using $(G-CR)$, we have

$$\begin{aligned}
||\mathfrak{b}_\eta - \mathfrak{a}^*||^2 &= ||\mathcal{Q}_{\mathfrak{B}_s}[(1-\kappa_\eta^2)\Upsilon_2\mathfrak{a}_\eta + \kappa_\eta^2\Upsilon_1\mathfrak{c}_\eta] - \mathcal{Q}_{\mathfrak{B}_s}[\mathfrak{a}^*]||^2 \\
&\leq ||(1-\kappa_\eta^2)(\Upsilon_2\mathfrak{a}_\eta - \mathfrak{a}^*) + \kappa_\eta^2(\Upsilon_1\mathfrak{c}_\eta - \mathfrak{a}^*)||^2 \\
&\leq (1-\kappa_\eta^2)||\Upsilon_2\mathfrak{a}_\eta - \mathfrak{a}^*||^2 + \kappa_\eta^2||\Upsilon_1\mathfrak{c}_\eta - \mathfrak{a}^*||^2 - \kappa_\eta^2(1-\kappa_\eta^2)\wp_c(||\Upsilon_2\mathfrak{a}_\eta - \Upsilon_1\mathfrak{c}_\eta||). \\
&\leq (1-\kappa_\eta^2)||\mathfrak{a}_\eta - \mathfrak{a}^*||^2 + \kappa_\eta^2||\mathfrak{c}_\eta - \mathfrak{a}^*||^2 - \kappa_\eta^2(1-\kappa_\eta^2)\wp_c(||\Upsilon_2\mathfrak{a}_\eta - \Upsilon_1\mathfrak{c}_\eta||) \\
&\leq ||\mathfrak{a}_\eta - \mathfrak{a}^*||^2 - \kappa_\eta^2(1-\kappa_\eta^2)\wp_c(||\Upsilon_2\mathfrak{a}_\eta - \Upsilon_1\mathfrak{c}_\eta||). \qquad (4.12)
\end{aligned}$$

Using inequality(4.11), we have

$$\begin{aligned}
&||\mathfrak{a}_{\eta+1} - \mathfrak{a}^*||^2 \\
&\leq \left[||\mathfrak{a}_\eta - \mathfrak{a}^*||^2 - \kappa_\eta^2(1-\kappa_\eta^2)\wp_c(||\Upsilon_2\mathfrak{a}_\eta - \Upsilon_1\mathfrak{c}_\eta||)\right] - \kappa_\eta^1(1-\kappa_\eta^1)\wp_c(||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta||) \\
&\leq \left[||\mathfrak{a}_\eta - \mathfrak{a}^*||^2 - \kappa_\eta^1\kappa_\eta^2(1-\kappa_\eta^2)\wp_c(||\Upsilon_2\mathfrak{a}_\eta - \Upsilon_1\mathfrak{c}_\eta||)\right] - \kappa_\eta^1(1-\kappa_\eta^1)\wp_c(||\mathfrak{b}_\eta - \Upsilon_1\mathfrak{b}_\eta||).
\end{aligned}$$

Noticeably $\underline{a}\ \underline{b}(1-\overline{b}) \leq \kappa_\eta^1\kappa_\eta^2(1-\kappa_\eta^2)\ \forall\ \eta \in \mathbb{N} \cup\ \{0\}$. We obtain that

$$\begin{aligned}
\underline{a}\ \underline{b}\sum_{i=0}^{\eta}\wp_c(||\Upsilon_2\mathfrak{a}_i - \Upsilon_1\mathfrak{c}_i||) &\leq ||\mathfrak{a}_0 - \mathfrak{a}^*||^2 - ||\mathfrak{a}_{\eta+1} - \mathfrak{a}^*||^2 \\
&\qquad - \sum_{i=0}^{\eta}\kappa_\eta^1(1-\kappa_\eta^1)\wp_c(||\mathfrak{b}_i - \Upsilon_1\mathfrak{b}_i||).
\end{aligned}$$

Now, we have

$$\underline{a}\ \underline{b}\sum_{\eta=0}^{\infty}\wp_c(||\Upsilon_2\mathfrak{a}_\eta-\Upsilon_1\mathfrak{c}_\eta||)\leq||\mathfrak{a}_0-\mathfrak{a}^*||^2<\infty.$$

It results in that

$$\lim_{\eta\to\infty}||\Upsilon_2\mathfrak{a}_\eta-\Upsilon_1\mathfrak{c}_\eta||=0.$$

Using the inequality (4.12), we have

$$\begin{aligned}
||\mathfrak{b}_\eta-\mathfrak{a}^*||&\leq(1-\kappa_\eta^2)||\mathfrak{a}_\eta-\mathfrak{a}^*||^2+\kappa_\eta^2\Big[||(1-\kappa_\eta^3)(\mathfrak{a}_\eta-\mathfrak{a}^*)-\kappa_\eta^3(\Upsilon_2\mathfrak{a}_\eta-\mathfrak{a}^*)||^2\Big]\\
&\quad-\kappa_\eta^2(1-\kappa_\eta^2)\wp_c(||\Upsilon_2\mathfrak{a}_\eta-\Upsilon_1\mathfrak{c}_\eta||)\\
&\leq(1-\kappa_\eta^2)||\mathfrak{a}_\eta-\mathfrak{a}^*||^2+\kappa_\eta^2\Big[(1-\kappa_\eta^3)||\mathfrak{a}_\eta-\mathfrak{a}^*||^2+\kappa_\eta^3||\Upsilon_2\mathfrak{a}_\eta-\mathfrak{a}^*)||^2\\
&\quad-\kappa_\eta^3(1-\kappa_\eta^3)\wp_c(||\mathfrak{a}_\eta-\Upsilon_2\mathfrak{a}_\eta||)\Big]-\kappa_\eta^2(1-\kappa_\eta^2)\wp_c(||\Upsilon_2\mathfrak{a}_\eta-\Upsilon_1\mathfrak{c}_\eta||).\\
&\leq||\mathfrak{a}_\eta-\mathfrak{a}^*||-\kappa_\eta^2\kappa_\eta^3(1-\kappa_\eta^2)\wp_c(||\mathfrak{a}_\eta-\Upsilon_2\mathfrak{a}_\eta||)-\kappa_\eta^2(1-\kappa_\eta^2)(||\Upsilon_2\mathfrak{a}_\eta-\Upsilon_1\mathfrak{c}_\eta||),
\end{aligned}$$

$\forall\ \eta\in\mathbb{N}\cup\ \{0\}$. On the other hand, from inequality (4.11), we have

$$\begin{aligned}
&||\mathfrak{a}_{\eta+1}-\mathfrak{a}^*||\\
&\quad=||\mathfrak{b}_\eta-\mathfrak{a}^*||^2-\kappa_\eta^1(1-\kappa_\eta^1)\wp_c(||\mathfrak{b}_\eta-\Upsilon_1\mathfrak{b}_\eta||\\
&\quad=\Big[||\mathfrak{a}_\eta-\mathfrak{a}^*||-\kappa_\eta^2\kappa_\eta^3(1-\kappa_\eta^2)\wp_c(||\mathfrak{a}_\eta-\Upsilon_2\mathfrak{c}_\eta||)-\kappa_\eta^2(1-\kappa_\eta^2)(||\Upsilon_2\mathfrak{a}_\eta-\Upsilon_1\mathfrak{c}_\eta||)\Big]\\
&\quad-\kappa_\eta^1(1-\kappa_\eta^1)\wp_c(||\mathfrak{b}_\eta-\Upsilon_1\mathfrak{b}_\eta||.
\end{aligned}$$

Therefore, $\underline{b}\ \underline{c}(1-\overline{c})\leq\mathfrak{b}_\eta\mathfrak{c}_\eta(1-\mathfrak{c}_\eta)$, $\forall\ \eta\in\mathbb{N}\cup\ \{0\}$. Noticeably

$$\begin{aligned}
\underline{b}\ \underline{c}(1-\overline{c})\sum_{i=0}^{\eta}\wp_c(\mathfrak{a}_i-\Upsilon_2\mathfrak{c}_i)\leq||\mathfrak{a}_0-\mathfrak{a}^*||^2-||\mathfrak{a}_{\eta+1}-\mathfrak{a}^*||^2&\\
-\sum_{i=0}^{\eta}\kappa_i^2(1-\kappa_i^2)\wp_c(||\Upsilon_2\mathfrak{a}_\eta-\Upsilon_1\mathfrak{c}_\eta||)&\\
-\sum_{i=0}^{\eta}\kappa_i^1(1-\kappa_i^1)\wp_c(||\mathfrak{b}_i-\Upsilon_1\mathfrak{b}_i||),&
\end{aligned}$$

which follows that $\lim_{\eta\to\infty}||\mathfrak{a}_\eta-\Upsilon_2\mathfrak{a}_\eta||\to0$. Note that

$$\begin{aligned}
||\mathfrak{c}_\eta-\mathfrak{a}_\eta||&=||\mathcal{Q}_{\mathfrak{B}_s}[(1-\kappa_\eta^3)\mathfrak{a}_\eta+\kappa_\eta^3\Upsilon_2\mathfrak{a}_\eta]-\mathcal{Q}_{\mathfrak{B}_s}[\mathfrak{a}^*]||\\
&=||\Upsilon_2\mathfrak{a}_\eta-\mathfrak{a}_\eta||\to0\text{ as }\eta\to\infty.
\end{aligned}$$

It is given that Υ_2 is uniformly continuous, so using Proposition (3.5)

$$\lim_{\eta\to\infty}||\mathfrak{c}_\eta-\Upsilon_2\mathfrak{c}_\eta||=0.$$

Therefore, from

$$\lim_{\eta\to\infty}||\Upsilon_2\mathfrak{a}_\eta-\Upsilon_1\mathfrak{c}_\eta||=0,$$

we have

$$||\mathfrak{a}_\eta - \Upsilon_2 \mathfrak{a}_\eta|| = 0.$$

(3) Let $\mathfrak{B}$ satisfies the Opial condition and Υ_1 and Υ_2 with CFP ω, where $\omega \in \mathcal{CB}_r[\mathfrak{a}^*] \cap \mathfrak{B}_s$. Lemma 4.1 results that $\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \omega||$ exists. Let $\exists$ $\{\mathfrak{a}_{\eta_p}\}$ and $\{\mathfrak{a}_{\theta_q}\}$ convergent to two distinct points ω_1 and ω_2 in $\mathcal{CB}_r \cap \mathfrak{B}_s$, respectively. Since both $I - \Upsilon_1$ and $I - \Upsilon_2$ are demiclosed at 0, we have

$$\Upsilon_1 \omega_1 = \Upsilon_2 \omega_1 = \omega$$

and

$$\Upsilon_1 \omega_2 = \Upsilon_2 \omega_2 = \omega.$$

Furthermore, the Opial condition results

$$\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \omega_1|| = \lim_{p\to\infty} ||\mathfrak{a}_{\eta_p} - \omega_1|| < \lim_{q\to\infty} ||\mathfrak{a}_{\theta_q} - \omega_2|| = \lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \omega_2||.$$

In similar manner, we have

$$\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \omega_2|| < \lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \omega_1||,$$

which is a contradiction. Hence, $\omega_1 = \omega_2$, which confirms the existence of the convergent sequence $\{\mathfrak{a}_\eta\}$ which converges weakly to $\omega \in \mathcal{F}_{(\Upsilon_1,\Upsilon_2)} \cap \mathcal{CB}_r[\mathfrak{a}^*]$. □

Also, if any nonexpansive mapping is uniformly continuous, we may deduce a convergence theorem for estimating the common FP_s of two nonexpansive mappings from Theorem 4.2 and Lemma 3.3.

Theorem 4.3. *Let $\emptyset \neq \mathfrak{B}_s \subseteq \mathfrak{B}$ with $\mathcal{Q}_{\mathfrak{B}_s}$ as the sunny nonexpansive retraction. Let $\Upsilon_1, \Upsilon_2 : \mathfrak{B}_s \to \mathfrak{B}$ be nonexpansive mappings such that $\mathcal{F}_{(\Upsilon_1,\Upsilon_2)} \neq \emptyset$. Let the real sequences $\{\kappa^1_\eta\}$, $\{\kappa^2_\eta\}$ and $\{\kappa^3_\eta\} \ni 0 < a \leq \kappa^1_\eta \leq \overline{a} < 1$, $0 < b \leq \kappa^2_\eta \leq \overline{b} < 1$ and $0 < c \leq \kappa^3_\eta \leq \overline{c} < 1$ $\forall$ $\eta \in \mathbb{N} \cup \{0\}$. Let $\mathfrak{a}_0 \in \mathfrak{B}_s$ and $\mathcal{P}_{\mathcal{F}_{(\Upsilon_1,\Upsilon_2)}}(\mathfrak{a}_0) = \mathfrak{a}^*$. Let $\{\mathfrak{a}_\eta\}$ be the sequence defined by $(G - CR)$. Then, we have*

1. *$\{\mathfrak{a}_\eta\}$ is in a closed convex bounded set $\mathcal{CB}_r[\mathfrak{a}^*] \cap \mathfrak{B}_s$, where*
$$r \in (0, \infty) \ni ||\mathfrak{a}_0 - \mathfrak{a}^*|| \leq r.$$
2. *$\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \Upsilon_1 \mathfrak{a}_\eta|| = 0$ and $\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - \Upsilon_2 \mathfrak{a}_\eta|| = 0$ with the same error bounds (2) defined in Theorem 4.2.*
3. *If $I - \Upsilon_2$ and $I - \Upsilon_1$ are demiclosed at 0 and $\mathfrak{B}$ satisfies the Opial condition, then $\{\mathfrak{a}_\eta\}$ is convergent to an element of $\mathcal{F}_{(\Upsilon_2,\Upsilon_1)} \cap \mathcal{CB}_r[\mathfrak{a}^*]$, where the convergence is weak convergence.*

We may restate condition (3) of Theorem 4.3 as if $\mathfrak{B}$ meets the Opial condition, $\{\mathfrak{a}_\eta\}$ weakly converges to an element of $\mathcal{F}_{(\Upsilon_1,\Upsilon_2)}$, if $\mathcal{P}_{\mathcal{F}_{(\Upsilon_1,\Upsilon_2)}}$ cannot be determined. Therefore we can define the following:

Corollary 4.4. *Let Υ_1, Υ_2 : $\mathcal{H}_{\mathfrak{s}_*} \to \mathcal{H}_{\mathfrak{s}_*}$ be nonexpansive mappings such that $\mathcal{F}_{(\Upsilon_1,\Upsilon_2)} \neq \emptyset$. Let the real sequences $\{\kappa^1_\eta\}$, $\{\kappa^2_\eta\}$ and $\{\kappa^3_\eta\} \ni 0 < a \leq \kappa^1_\eta \leq \overline{a} < 1$,*

$0 < b \leq \kappa_\eta^2 \leq \overline{b} < 1$ *and* $0 < c \leq \kappa_\eta^3 \leq \overline{c} < 1 \; \forall \; \eta \in \mathbb{N} \cup \; \{0\}$. *Let the sequence* $\{\mathfrak{a}_\eta\}$ *is defined as follows*

$$\begin{cases} \mathfrak{a}_0 \in \mathfrak{B}_s, \\ \mathfrak{a}_{\eta+1} = (1-\kappa_\eta^1)\mathfrak{b}_\eta + \kappa_\eta^1 \Upsilon_1 \mathfrak{b}_\eta, \\ \mathfrak{b}_\eta = (1-\kappa_\eta^2)\Upsilon_2 \mathfrak{a}_\eta + \kappa_\eta^2 \Upsilon_1 \mathfrak{c}_\eta, \\ \mathfrak{c}_\eta = (1-\kappa_{\eta^3})\mathfrak{a}_\eta + \kappa_{\eta^3} \Upsilon_2 \mathfrak{a}_\eta, \qquad \eta \in \mathbb{N} \cup 0. \end{cases} \tag{$CR-PPA$}$$

Then the sequence $\{\mathfrak{a}_\eta\}$ *is convergent weakly to an element of* $F_{(\Upsilon_1,\Upsilon_2)}$.

5. Application

It is important to note that various problems based on signal processing and machine learning can be expressed in accordance with the following manner.

Problem 1. For an $m-$accretive operator $\mathcal{A} : \mathfrak{B} \to 2^{\mathfrak{B}}$, find an element that satisfies

$$\mathfrak{a} \in \mathfrak{B} \text{ such that } 0 \in \mathcal{A}\mathfrak{a}. \tag{5.1}$$

PPA, introduced by Martinet (see [15], [14]) and generalized by Rockafellar ([17], [18]) is one of the popular methods to solve this problem. Also, Rockafellar [17] studied the weak convergence of the PPA, namely:

$$\mathfrak{a}_{\eta+1} = J_{\Delta_\eta}^{\mathcal{A}} \mathfrak{a}_\eta, \text{ for all } \eta \in \mathbb{N} \cup 0, \tag{5.2}$$

for the solution to Problem 5.1 and $\mathfrak{a}_0 \in \mathfrak{B}$. The weak and strong convergences of the sequence $\{x_\eta\}$ defined by equation (5.2) have been extensively studied in various ambient spaces e.g. Hilbert and Banach spaces (see [23], [22], [24], [25] and the references therein). The general form of Problem 1 is as follows:

Problem 2. Let the mappings $\mathcal{A}$, $\mathcal{A}_1 : \mathfrak{B} \to 2^{\mathfrak{B}}$ be $m-$accretive operators. Find an element

$$\mathfrak{a} \in \mathfrak{B} \ni \; 0 \in \mathcal{A}\mathfrak{a} \cap \mathcal{A}_1\mathfrak{a}, \tag{5.3}$$

when $\mathcal{A}$ and $\mathcal{A}_1$ are two maximal monotonic operators in a $\mathcal{H}_s$.

We are now eligible to utilize our observations, which are primarily focused on accretive operators' common zeros. We name $(G-CR)$ an iteration - based proximal point algorithm when $\Upsilon_1 = J_\Delta^{\mathcal{A}}$ and $\Upsilon_2 = J_\Delta^{\mathcal{A}_1}$. In a more generalized context, we now analyze its convergence to solve Problem 2.

Theorem 5.1. *Let* $\emptyset \neq \mathfrak{B}_s$ *be Opial condition. Let* $\mathcal{A} : Dom(\mathcal{A}) \subseteq \mathfrak{B}_s \to 2^{\mathfrak{B}}$ *and* $\mathcal{A}_1 : Dom(\mathcal{A}_1) \subseteq \mathfrak{B}_s \to 2^{\mathfrak{B}}$ *be accretive operators* $\ni Dom(\mathcal{A}) \subseteq \mathfrak{B}_s \subseteq \cap_{\lambda>0} Ran(I+\lambda\mathcal{A}), Dom(\mathcal{A}_1) \subseteq \mathfrak{B}_s \subseteq \cap_{\lambda>0} Ran(I+\lambda\mathcal{A}_1)$ *and* $\mathcal{A}^{-1}(0) \cap \mathcal{A}_1^{-1}(0) \neq \emptyset$. *Let* $\{\kappa_\eta^1\}$, $\{\kappa_\eta^2\}$, *and* $\{\kappa_\eta^3\}$ *be sequences of real numbers* $\ni 0 < a \leq \kappa_\eta^1 < \overline{a} < 1$, $b \leq \kappa_\eta^2 < \overline{b}$, *where* $b, \; \overline{b} \in (0,1)$ *and* $c \leq \kappa_\eta^3 < \overline{c}$, $c, \; \overline{c} \in (0,1) \; \forall \; \eta \in \mathbb{N} \cup 0$. *Let* $\Delta > 0$,

$\mathfrak{a}_0 \in \mathfrak{B}_s$ *and* $\mathcal{P}_{\mathcal{A}^{-1}(0)\cap\mathcal{A}_1 0^{-1}}(\mathfrak{a}_0) = \mathfrak{a}^*$. *Let the sequence* $\{\mathfrak{a}_\eta\}$ *be defined as follows:*

$$\begin{cases} \mathfrak{a}_0 \in \mathfrak{B}_s, \\ \mathfrak{a}_{\eta+1} = (1-\kappa^1_\eta)\mathfrak{b}_\eta + \kappa^1_\eta J^{\mathcal{A}}_\Delta \mathfrak{b}_\eta, \\ \mathfrak{b}_\eta = (1-\kappa^2_\eta) J^{\mathcal{A}_1}_\Delta \mathfrak{a}_\eta + \kappa^2_\eta J^{\mathcal{A}}_\Delta \mathfrak{c}_\eta, \\ \mathfrak{c}_\eta = (1-\kappa^3_\eta)\mathfrak{a}_\eta + \kappa^3_\eta J^{\mathcal{A}_1}_\Delta \mathfrak{a}_\eta, \end{cases}$$

Then, we have

1. $\{\mathfrak{a}_\eta\}$ *is in a closed convex bounded set* $\mathcal{CB}_r[\mathfrak{a}^*] \cap \mathfrak{B}_s$, *where*
$$r \in (0,\infty) \ni ||\mathfrak{a}_0 - \mathfrak{a}^*|| \leq r.$$
2. $\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - J^{\mathcal{A}}_\Delta \mathfrak{a}_\eta|| = 0$ *and* $\lim_{\eta\to\infty} ||\mathfrak{a}_\eta - J^{\mathcal{A}_1}_\Delta \mathfrak{a}_\eta|| = 0$ *with the same error bounds (2) defined in Theorem 4.2 where* $\Upsilon_1 = J^{\mathcal{A}}_\Delta$ *and* $\Upsilon_2 = J^{\mathcal{A}_1}_\Delta$.
3. $\{\mathfrak{a}_\eta\}$ *is convergent to an element of* $\mathcal{A}^{-1}(0)\cap\mathcal{A}_1^{-1}(0)\cap\mathcal{CB}[\mathfrak{a}^*]$ *and the convergence is weak convergence.*

Proof. As $Dom(\mathcal{A}) \subseteq \mathfrak{B}_s \subseteq \cap_{\lambda>0} Ran(I+\lambda\mathcal{A})$, it is to note that $J^{\mathcal{A}}_\Delta : \mathfrak{B}_s \to \mathfrak{B}_s$ is nonexpansive. Also, $J^{\mathcal{A}}_\Delta : \mathfrak{B}_s \to \mathfrak{B}_s$ is nonexpansive. Also, $Dom(\mathcal{A}) \cap Dom(\mathcal{B}) \subseteq \mathfrak{B}_s$, hence we have

$$\begin{aligned} \mathfrak{a} \in \mathcal{A}^{-1}(0)\mathcal{A}_1^{-1}(0) &\implies \mathfrak{a} \in Dom(\mathcal{A}) \cap Dom(\mathcal{A}_1) \text{ with } 0 \in \mathcal{A}\mathfrak{a} \text{ and } 0 \in \mathcal{A}_1\mathfrak{a} \\ &\implies \mathfrak{a} \in \mathfrak{B}_s \text{ with } J^{\mathcal{A}}_\Delta \mathfrak{a} = \mathfrak{a} \text{ and } J^{\mathcal{A}_1}_\Delta \mathfrak{a} = \mathfrak{a} \\ &\implies \mathfrak{a} \in F_{(J^{\mathcal{A}}_\Delta, J^{\mathcal{A}_1}_\Delta)}. \end{aligned} \tag{5.4}$$

Substitute $\Upsilon_1 = J^{\mathcal{A}}_\Delta$ and $\Upsilon_2 = J^{\mathcal{A}_1}_\Delta$. As a result, Theorem 5.1 refers to the proof from Theorem 4.3.

Example 5.2. For the problem given below, find the element which satisfies

$$\alpha \in \overset{\circ}{\mathcal{J}} := \partial\mathcal{A}^{-1}(0) \cap \partial\mathcal{A}_1^{-1}(0),$$

where $\mathcal{A}, \mathcal{A}_1 : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are defined as follows:

$$\mathcal{A}(\mathfrak{a}) = \frac{1}{2}\langle \nabla_f(\mathfrak{a}), \mathfrak{a})\rangle + \langle \mathfrak{a}, \beta\rangle.$$

Also

$$\mathcal{A}_1(\mathfrak{a}) = \frac{1}{2}\langle \nabla_g(\mathfrak{a}), \mathfrak{a})\rangle + \langle \mathfrak{a}, \gamma\rangle$$

$\forall\ \mathfrak{a} \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$ and

$$\nabla_f = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 2 & -3 \\ -1 & -1 & 3 \end{pmatrix}$$

and

$$\nabla_g = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$\beta = (2,6,8)$ and $\gamma = (2,6,0)$. Here, it easy to conclude that the functions ∇_f and ∇_g are convex and continuous as well on $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $del\nabla_f(\cdot) = \mathcal{A}(\cdot) + \beta$, $del\nabla_g(\cdot) = \mathcal{A}_1(\cdot) + \gamma$ and

$$\overset{\circ}{\mathcal{J}} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c} : \mathfrak{a} + \mathfrak{b} = 8,\ \mathfrak{c} = 0\}.$$

Let us define a sequence $\{\mathfrak{a}_\eta, \mathfrak{b}_\eta, \mathfrak{c}_\eta\}$ with initial value $\{\mathfrak{a}_0, \mathfrak{b}_0, \mathfrak{c}_0\}$ as follows:

$$\begin{cases} \mathfrak{a}_0 \in \mathfrak{B}_s, \\ (\mathfrak{a}^1_{\eta+1}, \mathfrak{b}^1_{\eta+1}, \mathfrak{c}^1_{\eta+1}) = (1-\kappa^1_\eta)(\mathfrak{b}^1_\eta, \mathfrak{b}^2_\eta, \mathfrak{b}^3_\eta) + \kappa^1_\eta \Upsilon_1(\mathfrak{b}^1_\eta, \mathfrak{b}^2_\eta, \mathfrak{b}^3_\eta), \\ (\mathfrak{b}^1_\eta, \mathfrak{b}^2_\eta, \mathfrak{b}^3_\eta) = (1-\kappa^2_\eta)\Upsilon_2(\mathfrak{a}^1_\eta, \mathfrak{b}^1_\eta, \mathfrak{c}^1_\eta) + \kappa^2_\eta \Upsilon_1(\mathfrak{c}^1_\eta, \mathfrak{c}^2_\eta, \mathfrak{c}^2_\eta), \\ (\mathfrak{c}^1_\eta, \mathfrak{c}^2_\eta, \mathfrak{c}^2_\eta) = (1-\kappa^3_\eta)(\mathfrak{a}^1_\eta, \mathfrak{b}^1_\eta, \mathfrak{c}^1_\eta) + \kappa^3_\eta \Upsilon_2(\mathfrak{a}^1_\eta, \mathfrak{b}^1_\eta, \mathfrak{c}^1_\eta), \end{cases} \tag{$\mathcal{E}$}$$

where $\Upsilon_1 = (I + del\nabla_f)^{-1}$ and $\Upsilon_2 = (I + del\nabla_g)^{-1}$, $0 < \kappa^1_\eta, \kappa^2_\eta, \kappa^3_\eta < 1$. Using initial value as $(\mathfrak{a}_0, \mathfrak{b}_0, \mathfrak{c}_0)$, $\forall\ \mathfrak{a}_0, \mathfrak{b}_0, \mathfrak{c}_0 \in \mathbb{R}$ in Theorem 4.2, we can find the solution for distinct values of $(\mathfrak{a}_0, \mathfrak{b}_0, \mathfrak{c}_0)$.

Conclusion. Inspired by two well-known concepts, $CR-$iterative algorithm by Chug et al. [5] and common zero of two accretive operators by Kim & Tuyen [10], in this analysis we have introduced the Generalized $G-CR$ iteration algorithm and analyzed its convergence behaviour to find CFP_s for nonself $QNEMs$ in convex Banach spaces. In order to understand the work, application of the the same is also analyzed.

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Generalized Szász-Mirakian type operators

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Abstract. In this paper we propose certain modifications of Szász-Mirakian type operators and study their approximation properties. We also give a Voronovskaya type theorem for these operators.

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1. Introduction

Classical Szász-Mirakian operator is defined as

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{(k)!} f\left(\frac{k}{n}\right) \tag{1.1}$$

with $x \in R_0 = [0,\infty), n \in N = \{1,2,3,\dots\}, k \in N_0 = N \cup 0$ and $f \in C(R_0)$. The approximation behaviour of this operator for bounded functions is well known (see, e.g. [2, 8]). Hermann considered this operator on a much wider class, growing faster than exponentially. In 2005, Schurer[6, 7] type generalization was given by Moreno [4] for this operator 1.1.

$$S_{n,p}(f;x) = e^{-(n+p)x} \sum_{k=0}^{\infty} \frac{((n+p)x)^k}{(k)!} f\left(\frac{k}{n}\right), x \in R_0, n \in N, p \in N_0.$$

Later Firlej and Rempulska [3] introduced a modified Szász-Mirakian operator:

$$\bar{S}_n(f;x) = \frac{1}{\cosh(nx)} \sum_{k=0}^{\infty} \frac{(nx)^{2k}}{(2k)!} f\left(\frac{2k}{n}\right), x \in R_0, n \in N.$$

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Motivated by the above two modifications now we consider Szász-Mirakian type operators for $f \in C_B$

$$\widehat{S}_{n,p}(f;x) = \frac{1}{\cosh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k}}{(2k)!} f\left(\frac{2k}{n}\right), x \in R_0, n \in N, p \in N_0. \quad (1.2)$$

and $coshx, tanhx$ are elementary hyperbolic functions. Let

$$C_B^2 = \{f \in C_B \cap C^2(R_0) : f'; f'' \in C_B\}$$

be the space of real-valued functions uniformly continuous and bounded on $R_0 = [0;\infty)$ and let the norm in C_B be given by the formula

$$\|f\| = \sup_{x \in R_0} |f(x)|.$$

In the year 2008, Deo et al. [1] studied Voronovskaya type results for modified Bernstein operators. Now the purpose of this study is to give Voronovskaya type theorems for Schurer ([6, 7]) as well Firlej and Rempulska [3] type modification of Szász-Mirakian operators.

2. Auxiliary results

In this section we prove some results on $\widehat{S}_{n,p}$ that will help in establishing the main result.

Lemma 2.1. *For each $n \in N$ and $x \in R_0$ we have*

$$\widehat{S}_{n,p}(1;x) = 1, \quad (2.1)$$

$$\widehat{S}_{n,p}(t;x) = \frac{(n+p)x}{n} \tanh((n+p)x), \quad (2.2)$$

$$\widehat{S}_{n,p}(t^2;x) = \frac{((n+p)x)^2}{n^2} + \frac{(n+p)x}{n^2} \tanh((n+p)x), \quad (2.3)$$

$$\widehat{S}_{n,p}(t^3;x) = \frac{1}{n^3}\left[\{((n+p)x)^3 + (n+p)x\} \tanh((n+p)x) + 3((n+p)x)^2\right], \quad (2.4)$$

and

$$\widehat{S}_{n,p}(t^4;x) = \frac{1}{n^4}\left[((n+p)x)^4 + \{6((n+p)x)^3 + (n+p)x\} \tanh((n+p)x) + 7((n+p)x)^2\right]. \quad (2.5)$$

Proof. From (1.2) we can easily obtain (2.1) and

$$\widehat{S}_{n,p}(t;x) = \frac{1}{\cosh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k}}{(2k)!} \left(\frac{2k}{n}\right) = \frac{(n+p)x}{n} \tanh((n+p)x).$$

$$\begin{aligned}
\widehat{S}_{n,p}(t^2;x) &= \frac{1}{\cosh((n+p)x)}\sum_{k=0}^{\infty}\frac{((n+p)x)^{2k}}{(2k)!}\left(\frac{2k}{n}\right)^2 \\
&= \frac{1}{\cosh((n+p)x)}\sum_{k=0}^{\infty}\frac{((n+p)x)^{2k}}{(2k)!}\frac{\{(2k)(2k-1)+2k\}}{n^2} \\
&= \frac{((n+p)x)^2}{n^2}+\frac{(n+p)x}{n^2}\tanh((n+p)x)
\end{aligned}$$

$$\begin{aligned}
\widehat{S}_{n,p}(t^3;x) &= \frac{1}{\cosh((n+p)x)}\sum_{k=0}^{\infty}\frac{((n+p)x)^{2k}}{(2k)!}\left(\frac{2k}{n}\right)^3 \\
&= \frac{1}{\cosh((n+p)x)}\sum_{k=0}^{\infty}\frac{((n+p)x)^{2k}}{(2k)!}\frac{\{(2k)(2k-1)(2k-2)+6k(2k-1)+2k\}}{n^3} \\
&= \frac{1}{\cosh((n+p)x)}\Big[\frac{((n+p)x)^3+(n+p)x}{n^3}\sinh((n+p)x) \\
&\quad+\frac{3((n+p)x)^2}{n^3}\cosh((n+p)x)\Big] \\
&= \frac{1}{n^3}\left[((n+p)x)^3\tanh((n+p)x)+3((n+p)x)^2+(n+p)x\tanh((n+p)x)\right].
\end{aligned}$$

$$\begin{aligned}
\widehat{S}_{n,p}(t^4;x) &= \frac{1}{\cosh((n+p)x)}\sum_{k=0}^{\infty}\frac{((n+p)x)^{2k}}{(2k)!}\left(\frac{2k}{n}\right)^4 \\
&= \frac{1}{\cosh((n+p)x)}\sum_{k=0}^{\infty}\frac{((n+p)x)^{2k}}{(2k)!n^4}\Big[(2k)(2k-1)(2k-2)(2k-3) \\
&\quad+6(2k)(2k-1)(2k-2)+7(2k)(2k-1)+2k\Big] \\
&= \frac{1}{n^4}\left[((n+p)x)^4+(6((n+p)x)^3+(n+p)x)\tanh((n+p)x)+7((n+p)x)^2\right].
\end{aligned}$$

□

Using above Lemma 2.1, we shall prove the following Lemma.

Lemma 2.2. *The following equalities hold for all $x \in R_0$ and $n \in N$:*

$$\widehat{S}_{n,p}(t-x;x) = \frac{(n+p)x}{n}\left[\tanh((n+p)x)-1\right]+\frac{px}{n}$$

$$\widehat{S}_{n,p}((t-x)^2;x) = \frac{(n+p)}{n}\left(2x^2-\frac{x}{n}\right)(1-\tanh(n+p)x)+\frac{(n+p)x+p^2x^2}{n^2}$$

$$\begin{aligned}
\widehat{S}_{n,p}((t-x)^3;x) \quad &= \frac{(n+p)}{n}tanh((n+p)x)\left[\frac{(n+p)^2}{n^2}x^3+\frac{x}{n^2}-\frac{3x^2}{n}+3x^3\right] \\
&\quad+\frac{3(n+p)^2x^2}{n^3}-\frac{3x(n+p)^2x^2}{n^2}-x^3
\end{aligned}$$

$$\widehat{S}_{n,p}((t-x)^4;x)$$
$$= \frac{(n+p)}{n}\tanh((n+p)x)\Big[\frac{6(n+p)^2x^3}{n^3} - \frac{4(n+p)^2x^3}{n^2} + \frac{x}{n^3} - \frac{4x^2}{n^2} + \frac{6x^3}{n} - 4x^4\Big]$$
$$+ \frac{((n+p)x)^4 + 7((n+p)x)^2 - 12xn((n+p)x)^2 + 6x^2n^2((n+p)x)^2 + x^4n^4}{n^4}$$

Proof. Using Lemma 2.1 we have,

$$\widehat{S}_{n,p}(t-x;x) = \frac{(n+p)x}{n}\tanh((n+p)x) - x$$
$$= \frac{(n+p)x}{n}[\tanh((n+p)x) - 1] + \frac{px}{n} \tag{2.6}$$

$$\widehat{S}_{n,p}((t-x)^2;x) = \frac{((n+p)x)^2}{n^2} + \frac{(n+p)x}{n^2}\tanh((n+p)x)$$
$$- 2x\frac{(n+p)x}{n}\tanh((n+p)x) + x^2$$
$$= \tanh((n+p)x)\left[\frac{(n+p)x}{n^2} - 2x\frac{(n+p)x}{n}\right] + x^2 + \frac{((n+p)x)^2}{n^2} \tag{2.7}$$
$$= \frac{(n+p)}{n}\left(2x^2 - \frac{x}{n}\right)(1 - \tanh(n+p)x) + \frac{(n+p)x + p^2x^2}{n^2}$$

$$\widehat{S}_{n,p}((t-x)^3;x)$$
$$= \frac{1}{n^3}\left[((n+p)x)^3\tanh((n+p)x) + 3((n+p)x)^2 + (n+p)x\tanh((n+p)x)\right]$$
$$- 3x\left[\frac{((n+p)x)^2}{n^2} + \frac{(n+p)x}{n^2}\tanh((n+p)x)\right]$$
$$+ 3\frac{(n+p)x^3}{n}\tanh((n+p)x) - x^3 \tag{2.8}$$
$$= \frac{(n+p)}{n} tanh((n+p)x)\left[\frac{(n+p)^2}{n^2}x^3 + \frac{x}{n^2} - \frac{3x^2}{n} + 3x^3\right]$$
$$+ \frac{3(n+p)^2x^2}{n^3} - \frac{3x(n+p)^2x^2}{n^2} - x^3$$

$$\widehat{S}_{n,p}((t-x)^4;x) = \widehat{S}_{n,p}(t^4;x) - 4x\widehat{S}_{n,p}(t^3;x) + 6x^2\widehat{S}_{n,p}(t^2;x) - 4x^3\widehat{S}_{n,p}(t;x) + x^4\widehat{S}_{n,p}(1;x)$$
$$= \tanh((n+p)x)\Big[\frac{(n+p)x + 6((n+p)x)^3}{n^4} - \frac{4x\{((n+p)x)^3 + (n+p)x\}}{n^3}$$
$$+ \frac{6x^3(n+p) - 4x^4n(n+p)}{n^2}\Big]$$
$$+ \frac{((n+p)x)^4 + 7((n+p)x)^2 - 12xn((n+p)x)^2 + 6x^2n^2((n+p)x)^2 + x^4n^4}{n^4}$$
$$= \frac{(n+p)}{n}\tanh((n+p)x)\Big[\frac{6(n+p)^2x^3}{n^3} - \frac{4(n+p)^2x^3}{n^2} + \frac{x}{n^3} - \frac{4x^2}{n^2} + \frac{6x^3}{n} - 4x^4\Big]$$

$$+\frac{((n+p)x)^4+7((n+p)x)^2-12xn((n+p)x)^2+6x^2n^2((n+p)x)^2+x^4n^4}{n^4} \tag{2.9}$$

□

In order to prove a Voronovskaya type theorem we need the following lemmas.

Lemma 2.3. *For $n, r \in N, p \in N_0$ and $x \geq 0$, the following results hold*

$$(a)\ 0 \leq x^r(1-tanh(n+p)x) \leq 2^{1-r}r!(n+p)^{-r}$$

$$(b)\ \lim_{n\to\infty}\left[n\left\{\tanh((n+p)x)-1\right\}\right]=0$$

$$(c)\ \lim_{n\to\infty}\left\{\tanh((n+p)x)\right\}=1,$$

and

$$(d)\ \lim_{n\to\infty}\frac{\{\tanh((n+p)x)-1\}}{n}=0$$

Proof. We shall use an inequality considered in (eq.(22), [5]) which says that, for $m, r \in N$ and $x \geq 0$,

$$0 \leq x^r(1-tanhmx) \leq 2^{1-r}r!m^{-r}$$

Now on replacing m by $n+p, n \in N, p \in N_0$ we get the desired result.
Following the technique used in Lemma 2 of [5],we easily obtain (b),(c) and (d). □

Lemma 2.4. *For every fixed $x \in R_0$, we have*

$$(i)\ \lim_{n\to\infty} n\widehat{S}_{n,p}(t-x;x)=px,$$

$$(ii)\ \lim_{n\to\infty} n\widehat{S}_{n,p}((t-x)^2;x)=x,$$

Proof. Using Lemma 2.2 we obtain,

$$\begin{aligned}(i)\widehat{S}_{n,p}(t-x;x) &= \frac{(n+p)x}{n}\tanh((n+p)x)-x\\ &= \frac{(n+p)x}{n}\left[\tanh((n+p)x)-1\right]+\frac{px}{n}\\ &= x\left[\tanh((n+p)x)-1\right]+\frac{px}{n}\tanh((n+p)x).\end{aligned}$$

therefore

$$n\widehat{S}_{n,p}(t-x;x)=xn\left[\tanh((n+p)x)-1\right]+px\tanh((n+p)x).$$

Using Lemma 2.3 we get

$$\begin{aligned}\lim_{n\to\infty} n\widehat{S}_{n,p}(t-x;x) &= \lim_{n\to\infty}\left[xn\left\{\tanh((n+p)x)-1\right\}+px\tanh((n+p)x)\right]\\ &= px.\end{aligned}$$

$$(ii)\widehat{S}_{n,p}((t-x)^2;x)=\frac{(n+p)}{n}\left(2x^2-\frac{x}{n}\right)(1-\tanh(n+p)x)+\frac{(n+p)x+p^2x^2}{n^2}$$

$$\begin{aligned} n\widehat{S}_{n,p}((t-x)^2;x) &= 2x^2[n\{1-\tanh((n+p)x)\}] \\ &\quad + 2px^2\left[1-\{\tanh((n+p)x)\}\right] - x\left[\{1-\tanh((n+p)x)\}\right] \\ &\quad - \frac{px}{n}\left[1-\{\tanh((n+p)x)\}\right] + x + \frac{px}{n} + \frac{p^2x^2}{n} \end{aligned}$$

Again using Lemma 2.3 we obtain

$$\lim_{n\to\infty} n\widehat{S}_{n,p}((t-x)^2;x) = x \qquad \square$$

Lemma 2.5. *The following inequalities are satisfied for all $n \in N, p \in N_0$ and $x \in R_0$:*

$$|\widehat{S}_{n,p}(t-x;x)| \le \frac{px+1}{n}$$

$$|\widehat{S}_{n,p}((t-x)^2;x)| \le \frac{3+(p+1)x+p^2x^2}{n}$$

$$|\widehat{S}_{n,p}((t-x)^4;x)| \le \frac{47+(1+p)x+(3+10p+7p^2)x^2+6p^2x^3+p^4x^4}{n^2} \tag{2.10}$$

Proof. For $n, r \in N, p \in N_0$ *and* $x \ge 0$ we have

$$0 \le x^r(1-tanh(n+p)x) \le 2^{1-r}r!(n+p)^{-r}.$$

So from Lemma 2.2,

$$\begin{aligned} |\widehat{S}_{n,p}(t-x;x)| &= \left|\frac{(n+p)x}{n}\left[\tanh((n+p)x)-1\right]\right| + \frac{px}{n} \\ &\le \frac{px+1}{n} \end{aligned}$$

$$\begin{aligned} \widehat{S}_{n,p}((t-x)^2;x) &= \frac{((n+p)x)^2}{n^2} + \frac{(n+p)x}{n^2}\tanh((n+p)x) \\ &\quad - 2x\frac{(n+p)x}{n}\tanh((n+p)x) + x^2 \\ &= \tanh((n+p)x)\left[\frac{(n+p)x}{n^2} - 2x\frac{(n+p)x}{n}\right] + x^2 + \frac{((n+p)x)^2}{n^2} \\ &= (\tanh(n+p)x-1)\left[\frac{(n+p)x)}{n^2} - 2x\frac{(n+p)x}{n}\right] + \frac{(n+p)x+p^2x^2}{n^2} \end{aligned}$$

$$\begin{aligned} |\widehat{S}_{n,p}((t-x)^2;x)| &= \left|(\tanh(n+p)x-1)\left[\frac{(n+p)x)}{n^2} - 2x\frac{(n+p)x}{n}\right] + \frac{(n+p)x+p^2x^2}{n^2}\right| \\ &\le \frac{1+(n+p)x+p^2x^2}{n^2} + \frac{2}{n(n+p)} \\ &\le \frac{3+px+p^2x^2}{n^2} + \frac{x}{n} \\ &\le \frac{3+px+x+p^2x^2}{n} \\ &\le \frac{3+(p+1)x+p^2x^2}{n} \end{aligned}$$

$$
\begin{aligned}
&\widehat{S}_{n,p}((t-x)^4;x)\\
&= \frac{(n+p)}{n}\tanh((n+p)x)\Big[\frac{6(n+p)^2x^3}{n^3} - \frac{4(n+p)^2x^3}{n^2} + \frac{x}{n^3} - \frac{4x^2}{n^2} + \frac{6x^3}{n} - 4x^4\Big]\\
&\quad + \frac{((n+p)x)^4 + 7((n+p)x)^2 - 12xn((n+p)x)^2 + 6x^2n^2((n+p)x)^2 + x^4n^4}{n^4}\\
&= \tanh((n{+}p)x)\Big[\frac{6(n{+}p)^3x^3}{n^4} - \frac{4(n{+}p)^3x^4}{n^3} + \frac{(n{+}p)x}{n^4} - \frac{4(n+p)x^2}{n^3} + \frac{6(n+p)x^3}{n^2}\\
&- 4\frac{(n+p)}{n}x^4\Big] + \frac{(n+p)^4x^4}{n^4} + \frac{7(n+p)^2x^2}{n^4} - \frac{12(n+p)^2x^3}{n^3} + \frac{6(n+p)^2x^4}{n^2} + x^4\\
&= \Big(\tanh((n+p)x) - 1\Big)\Big[\frac{6(n+p)^3x^3}{n^4} - \frac{4(n+p)^3x^4}{n^3} + \frac{(n+p)x}{n^4} - \frac{4(n+p)x^2}{n^3}\\
&+ \frac{6(n{+}p)x^3}{n^2} - 4\frac{(n{+}p)}{n}x^4\Big] + \left(\frac{(n{+}p)^4}{n^4} - \frac{4(n{+}p)^3}{n^3} + \frac{6(n+p)^2}{n^2} - 4\frac{(n+p)}{n} + 1\right)x^4\\
&+ \Big(\frac{6(n+p)^3}{n^4} + \frac{6(n+p)}{n^2} - \frac{12(n+p)^2}{n^3}\Big)x^3 + \Big(\frac{7(n+p)^2}{n^4} - \frac{4(n+p)}{n^3}\Big)x^2 + \frac{(n+p)}{n^4}x
\end{aligned}
$$

$$
\begin{aligned}
|\widehat{S}_{n,p}((t-x)^4;x)| &\le \frac{10}{n^4} + \frac{16}{n^3(n+p)} + \frac{9}{n^2(n+p)^2} + \frac{12}{n(n+p)^3} +\\
&\quad \frac{p^4x^4}{n^4} + \frac{6p^2x^3}{n^3} + \frac{(3+10p+7p^2)x^2}{n^2} + \frac{(1+p)x}{n^2}\\
&\le \frac{47}{n^4} + \frac{p^4x^4 + 6p^2x^3 + (3+10p+7p^2)x^2 + (1+p)x}{n^2}\\
&\le \frac{47 + (1+p)x + (3+10p+7p^2)x^2 + 6p^2x^3 + p^4x^4}{n^2}
\end{aligned}
$$

□

3. Voronovskaya type theorems

In this section we give a Voronovskaya-type theorem for the operators $\widehat{S}_{n,p}$ with the help of properties of $\widehat{S}_{n,p}$, which are already mentioned in the above lemmas.

Lemma 3.1. *Suppose that x_0 is a fixed point in R_0 and $\varphi(t;x_0)$is a given function belonging to C_B and such that*

$$\lim_{t\to x_0} \varphi(t;x_0) = 0 \left(\lim_{t\to p} \varphi(t;0) = 0\right)$$

Then for a fixed $p \in N$

$$\lim_{n\to\infty} \widehat{S}_{n,p}(\varphi(t;x_0);x_0) = 0. \tag{3.1}$$

Proof. By (1.2) we have for $n \in N$ and a fixed point $x_0 \ge 0$

$$\widehat{S}_{n,p}(\varphi(t;x_0);x_0) = \frac{1}{cosh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k}}{(2k)!} \varphi\left(\frac{2k}{n};x_0\right) \tag{3.2}$$

Choose $\epsilon > 0$. Since $\varphi(\cdot; x_0) \in C_B$, there exists a positive constant $\delta \equiv \delta(\epsilon)$ such that

$$|\varphi(t; x_0)| < \frac{\epsilon}{2}, \quad \text{if } |t - x_0| < \delta, \ t \geq 0$$

Moreover there exists a positive constant M such that $|\varphi(t; x_0)| \leq M$ for all $t > 0$. Hence, from (3.2) we get for every $n \in N$

$$\begin{aligned} |\widehat{S}_{n,p}(\varphi(t; x_0); x_0)| &\leq \frac{1}{cosh((n+p)x)} \sum_{k \in Q_{1,n}} \frac{((n+p)x)^{2k}}{(2k)!} \left| \varphi\left(\frac{2k}{n}; x_0\right) \right| \\ &+ \frac{1}{cosh((n+p)x)} \sum_{k \in Q_{2,n}} \frac{((n+p)x)^{2k}}{(2k)!} \left| \varphi\left(\frac{2k}{n}; x_0\right) \right| \\ &= E_1 + E_2 \end{aligned} \tag{3.3}$$

where

$$Q_{1,n} = k \in N_0 : \left| \frac{2k}{n} - x_0 \right| < \delta$$

and

$$Q_{2,n} = k \in N_0 : \left| \frac{2k}{n} - x_0 \right| \geq \delta.$$

From (2.1) we get,

$$E_1 < \frac{\epsilon}{2} \frac{1}{cosh((n+p)x)} \sum_{k=0}^{\infty} \frac{((n+p)x)^{2k}}{(2k)!} = \frac{\epsilon}{2} \tag{3.4}$$

$$E_2 \leq M \frac{1}{cosh((n+p)x)} \sum_{k \in Q_{2,n}} \frac{((n+p)x)^{2k}}{(2k)!} \tag{3.5}$$

Since $|\frac{2k}{n} - x_0| \geq \delta$ implies $1 \leq \delta^{-2} \left(\frac{2k}{n} - x_0\right)^2$, we can write

$$\begin{aligned} E_2 &\leq M\delta^{-2} \frac{1}{cosh((n+p)x))} \sum_{k \in Q_{2,n}} \frac{((n+p)x)^{2k}}{(2k)!} \left(\frac{2k}{n} - x_0\right)^2 \\ &\leq M\delta^{-2} \widehat{S}_{n,p}((t - x_0)^2; x_0) \end{aligned}$$

which by Lemma 2.5 gives,

$$E_2 \leq \frac{M(3 + px_0 + x_0 + p^2 x_0^2)}{n\delta^2}$$

It is obvious that for given $\epsilon > 0, \delta > 0, M > 0$ *and* $x_0 \geq 0$ we can choose $n_0 \equiv n_0(\epsilon; \delta; M; x_0) \in N$ such that for all natural numbers $n > n_0$ one gets

$$\frac{M(3 + px_0 + x_0 + p^2 x_0^2)}{n\delta^2} < \frac{\epsilon}{2}$$

Hence,

$$E_2 < \frac{\epsilon}{2} \ for \ n > n_0. \tag{3.6}$$

Using equations (3.4) and(3.6) to (3.3) we get

$$\lim_{n \to \infty} \widehat{S}_{n,p}(\varphi(t; x_0); x_0) = 0$$

and this completes the proof. □

Theorem 3.2. *If* $f \in C_B^2$*, then for every fixed* $x \in R_0$ *one gets*

$$\lim_{n\to\infty} n\{\widehat{S}_{n,p}(f;x) - f(x)\} = pxf'(x) + \frac{x}{2}f''(x). \tag{3.7}$$

Proof. Let $x_0 \in R_0$ be a fixed point.Then by Taylor formula we can write for every $t \in R_0$,

$$f(t) = f(x_0) + f'(x_0)(t - x_0) + \frac{1}{2}f''(x_0)(t - x_0)^2 + \psi(t;x_0)(t - x_0)^2, \tag{3.8}$$

where $\psi(t;x_0) \in C_B$ and $\lim\limits_{t\to x_0} \psi(t;x_0) = \left(\lim\limits_{t\to 0+} \psi(t;0) = 0\right)$

On applying the operator $\widehat{S}_{n,p}$ on both sides of (3.8), we have for every $n \in N$,

$$\begin{aligned}\widehat{S}_{n,p}(f;x_0) - f(x_0) = f'(x_0)\widehat{S}_{n,p}(t - x_0;x_0) + \frac{1}{2}f''(x_0)\widehat{S}_{n,p}((t - x_0)^2;x_0)\\ + \widehat{S}_{n,p}(\psi(t;x_0)(t - x_0)^2;x_0)\end{aligned} \tag{3.9}$$

In view of (3.9) and by Holder's inequality we get for $n \in N$

$$|\widehat{S}_{n,p}(\psi(t;x_0)(t - x_0)^2;x_0)| \le \left\{\widehat{S}_{n,p}(\psi^2(t;x_0);x_0)\right\}^{1/2} \left\{\widehat{S}_{n,p}((t - x_0)^4;x_0)\right\}^{1/2} \tag{3.10}$$

Since the function $\varphi(t;x_0) = \psi^2(t;x_0), t \ge 0$ satisfies the assumption of Lemma 3.1 we have,

$$\lim_{n\to\infty} \widehat{S}_{n,p}(\psi^2(t;x_0);x_0) = 0$$

From (2.10) it follows that there exists a constant $M_1 = M_1(p,x_0)$ depending on p *and* x_0 such that

$$n^2\widehat{S}_{n,p}((t - x_0)^4;x_0) \le M_1 \ for \ n \in N$$

Consequently we obtain

$$\lim_{n\to\infty} n\widehat{S}_{n,p}(\psi(t;x_0)(t - x_0)^2;x_0) = 0 \tag{3.11}$$

from (3.10). Using Lemma 2.4 and (3.11), we derive immediately (3.7) from (3.9). Thus the proof is complete. □

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Global existence and uniqueness for viscoelastic equations with nonstandard growth conditions

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Abstract. This paper is devoted to the study of generalized viscoelastic nonlinear equations with Dirichlet-Neumann boundary conditions. We establish the local and uniqueness of weak solutions results in Sobolev spaces with variable exponents. Solutions are constructed as a limit of approximate solutions by a method independent of a compactness argument. We also discuss the global existence of solutions in the energy space.

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1. Introduction

In this paper, we study the global existence and uniqueness of weak solutions for the nonlinear viscoelastic equation with the $m(x)$-Laplacian operator

$$\begin{cases} u_{tt} - \Delta_{m(x)} u + w_1 \Delta^2 u(t) - w_2 \Delta u_t(t) + \alpha(t) \int_0^t \beta(t-s)\, \Delta u(s)\, \mathrm{d}s \\ \qquad + |u|^{p(x)-2} u(t) + \lambda g(u_t(t)) = b f(u(t)) \text{ in } \Omega \times \mathbb{R}^+, \\ \qquad u = \partial_\eta u = 0 \text{ on } \Gamma \times [0, +\infty[\,, \\ \qquad u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \text{ in } \Omega, \end{cases} \tag{1.1}$$

where $\Delta_{m(x)} u = div\left(|\nabla u|^{m(x)-2} \nabla u\right)$ is called the $m(x)$-Laplacian operator, $m(x)$ and $p(x)$ are two continuous functions on Ω, Ω is a bounded open subset of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma$, β is a memory kernel that decays exponentially,

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$g(u_t)$ is a nonlinear damping term, $f(u)$ is a nonlinear generalized source term, u_0 and u_1 are given functions, and ∂_η denotes the normal derivative directed outside of Ω and $Q = \Omega\times[0,T]$. Problem (1.1), with its general memory term $\alpha(t)\int_0^t \beta(t-s)\,\Delta u(s)ds$, can be regarded as a fourth order viscoelastic plate equation with a lower–order perturbation of the usual m-Laplacian type ($m(x) = const \geq 2$). It can also be regarded as an elastoplastic flow equation with some kind of memory effect. We note that for viscoelastic plate equations, it is usual to consider a memory of the general form $\alpha(t)\int_0^t \beta(t-s)\,\Delta^2 u(s)ds$. However, because the main dissipation of the system (1.1) is given by strong damping $-\Delta u_t(t)$, here we consider a weaker memory, acting only on $\Delta u(t)$. There is a large body of literature about the stability and global existence of viscoelasticity. We refer the reader to, [9, 10, 8, 18, 19, 4, 2, 3, 1]. Our objective in the present work is to extend the results established in the study of the differential equation about global existence with standard m-growth in the study of generalized problem (1.1) with nonstandard $m(x)$-growth. Equations with nonstandard growth occur in the mathematical modeling of various physical phenomena, for example, the flows of electrorheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, processes of filtration through porous media and image processing.

2. Literature overview and new contributions

The semilinear case with the classical Laplace operator (when $m(x) = m = const$) and when ($p(x) = p = const$), was studied by many authors. Other related works include:

1. The asymptotic behavior of solutions of the equations of linear viscoelasticity at large times was considered first by Dafermos [9] in 1970, where the general decay was discussed.
$$u_{tt} - \Delta^2 u(t) - \Delta u_t(t) + \int_0^t \beta(t-s)\,\Delta u(s)\,\mathrm{d}s = 0.$$
From a physical point of view, this type of problem usually arises in viscoelasticity.
2. With the usual m–Laplacian operator $m(x) = p(p = \text{const} \geq 2)$, a more general problem concerning the energy decay for a class of plate equations with memory and lower order perturbation of the p–Laplacian type
$$u_{tt} - div\left(|\nabla u|^{p-2}\nabla u\right) + \Delta^2 u(t) - \Delta u_t(t) + \int_0^t \beta(t-s)\,\Delta u(s)\,\mathrm{d}s + f(u(t)) = 0$$
has been extensively studied in [5].
3. Problem (1.1) without the viscoelastic term, with the usual m-Laplacian operator ($m(x) = m-1$), ($p = const \geq 2$) has been extensively studied by Yang et al [6, 7] concerning existence, nonexistence and long-term dynamics,
$$u_{tt} - div\left(|\nabla u|^{m-1}\nabla u\right) + \Delta^2 u(t) - \Delta u_t(t) + g(u_t(t)) + h(u(t)) = f(x,t)$$

4. The following problem:

$$u_{tt} - \Delta u(t) + \int_0^t \beta(t-s)\,\Delta(u(s,x))\mathrm{d}s + |u|^{p-2}u + \sigma(x)\,u_t = 0$$

for $\sigma\ :\Omega \to \mathbb{R}^+$, a function, which may be null on a part of the domain Ω, has been considered and studied by many authors [8].

By assuming $\sigma(x) > \sigma_0$ on the subdomain $\overline{\varpi} \subset \Omega$, the authors obtained an exponential rate of decay, provided that the kernel β satisfies:

$$\begin{cases} -\zeta_1\beta(t) \le \beta'(t) \le -\zeta_2\beta(t),\ t \ge 0, \\ \|\beta\|_{L^\infty(0,+\infty)} \text{ is small enough.} \end{cases}$$

Motivated by previous works, the goal of this paper is to establish the local and uniqueness of weak solution results in Sobolev spaces with variable exponents. We also discuss the global existence of solutions in the energy space. We pay specific properties caused by the variable exponents $m(.)$ and $p(.)$.

3. Problem statement

In this section we list and recall some well-known results and facts from the theory of Sobolev spaces with variable exponents. (For the details see [11, 12, 13, 14, 15]). Throughout the rest of the paper we assume that Ω is a bounded domain of $\mathbb{R}^n$, $n \ge 2$ with smooth boundary Γ and assume that $p(x)$ and $m(x)$ satisfy:

$$\begin{cases} 2 < p_- \le p(x) \le p_+ < p_*(x) < \infty, \\ 2 < m_- \le m(x) \le m_+ < m_*(x) < \infty \end{cases} \tag{3.1}$$

where

$$\varphi_+ = ess \sup_{x\in\Omega} \varphi(x),\ \ \varphi_- = ess \inf_{x\in\Omega} \varphi(x)$$

and

$$\varphi_*(x) \le \begin{cases} \dfrac{n\varphi(x)}{(n-\varphi(x))_+}, \text{ if } \varphi_+ < n \\ +\infty, \text{ if } \varphi_+ \ge n. \end{cases} \tag{3.2}$$

We also assume that

$$|m(x) - m(y)| \le \frac{M}{|\log|x-y||}, \text{ for all } x,\ y \text{ in } \Omega \text{ with } |x-y| < \frac{1}{2}, \tag{3.3}$$

with $M > 0$ and

$$m_* > ess \sup_{\{x\in\Omega\}} m(x) \tag{3.4}$$

Let $p\ :\ \Omega \to [1,\infty]$ be a measurable function. We denote by $L^{p(.)}(\Omega)$ the set of measurable functions u on Ω such that

$$A_{p(.)}(u) = \int_{\{x\in\Omega|p(x)<\infty\}} |u(x)|^{p(x)}\mathrm{d}x + \underset{\{x\in\Omega|p(x)=\infty\}}{ess\ \sup} |u(x)| < \infty$$

The set $L^{p(.)}(\Omega)$ equipped with the Luxemburg norm

$$\|u\|_{p(.)} = \|u\|_{L^{p(.)}(\Omega)} = \inf\left\{\mu > 0,\ A_{p(.)}\left(\frac{u}{\mu}\right) \leq 1\right\},$$

is a Banach space with

$$\min\left(\|u\|_{p(.)}^{p_-}, \|u\|_{p(.)}^{p_+}\right) \leq A_{p(.)}(u) \leq \max\left(\|u\|_{p(.)}^{p_-}, \|u\|_{p(.)}^{p_+}\right)$$

and the generalized Hölder's inequality holds.

Let p satisfy the following Zhikov–Fan uniform local continuity condition :

$$|p(x) - p(y)| \leq \frac{M}{|\log|x-y||}, \text{ for all } x,\ y \text{ in } \Omega \text{ with } |x-y| < \frac{1}{2},\ M > 0, \tag{3.5}$$

with $\underset{\{x\in\Omega\}}{ess\ \inf}(p^*(x) - p(x)) > 0$.

- If condition (3.5) is fulfilled, Ω has a finite measure and p, q are variable exponents so that $p(x) \leq q(x)$ almost everywhere in Ω, then the embedding $L^{q(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous.
- If $p : \Omega \to [1, +\infty)$ is a measurable function and $p_* > \underset{\{x\in\Omega\}}{ess\ \sup} p(x)$ with $p_* \leq \frac{2n}{n-2}\,(n > 2)$, $\left(p_* \leq \frac{2n}{n-4}\,(n > 4)\right)$, then the embeddings $H_0^1(\Omega) \hookrightarrow L^{p(.)}(\Omega)$, and $\left(H_0^2(\Omega) \hookrightarrow L^{p(.)}(\Omega)\right)$ are continuous and compact respectively.

Let us state the precise hypotheses on g, f, α and β :
α is a measurable nonincreasing differentiable bounded function on $\mathbb{R}^+$ and

$$\alpha_+ \geq \alpha(0) \geq \alpha(t) > 0,\ t \geq 0. \tag{3.6}$$

Let g be increasing C^1–function such that:

$$\left\{\begin{array}{r} xg(x) \geq d_0 |x|^{\sigma(x)},\ x \in \mathbb{R}, \\ |g(x)| \leq d_1|x| + d_2|x|^{\sigma(x)-1},\ x \in \mathbb{R},\ d_i \geq 0, \\ 2 < \sigma_- \leq \sigma(x) \leq \sigma_+ \leq p(x) \leq p_+ \leq \dfrac{2n}{n-2} < \infty,\ n \geq 3. \end{array}\right. \tag{3.7}$$

Let $f(x,s) \in C^1(\Omega \times \mathbb{R})$ satisfy:

$$sf(x,s) + k_1(x)|s| \geq p(x)\widehat{f}(x,s), \tag{3.8}$$

and the growth conditions

$$\left\{\begin{array}{r} |f(x,s)| \leq l_1\left(|s|^{\theta} + k_2(x)\right); \\ |f_s(x,s)| \leq l_1\left(|s|^{\theta-1} + k_3(x)\right) \text{ in } \Omega \times \mathbb{R},\ \text{and } 1 < \theta \leq \dfrac{p_-}{2}, \end{array}\right. \tag{3.9}$$

where $\widehat{f}(x,s) = \widehat{f}(s) = \int_0^s f(x,\zeta)\,d\zeta$, with some $l_0,\ l_1 > 0$ and the nonnegative functions $k_1(x), k_2(x), k_3(x) \in L^\infty(\Omega)$, a.e. $x \in \Omega$. .

The memory kernel $\beta : [0, +\infty[\to [0, +\infty[$ is a differentiable bounded function such that

$$\begin{cases} \beta(0) = \beta_0 > 0, \ \ \infty > \displaystyle\int_0^\infty \beta(t)\mathrm{d}t = \beta_1; \\ \qquad\qquad w_1\lambda_1 - \alpha(0)\beta_1 > 0; \\ \alpha(t)\beta(t) + \alpha'(t)\displaystyle\int_0^t \beta(s)\,\mathrm{d}s \geq 0 \ \ t \in \mathbb{R}^+. \end{cases} \tag{3.10}$$

there exists $K > 0$ such that

$$\beta'(t) \leq -K\beta(t) \quad \forall t \geq 0. \tag{3.11}$$

where $\lambda_1 > 0$ is determined by the imbedding inequality

$$\lambda_1 |\nabla u(t)|^2 \leq |\Delta u|^2 . \tag{3.12}$$

Remark 3.1. Typical examples of functions satisfying (3.10) and (3.11), are

$$\beta(t) = \beta_0 e^{-at}, \quad a \geq \max\left(\frac{\beta_0\alpha(0)}{w_1\lambda_1}, K\right);$$

$$\alpha(t) = \alpha(0)\, e^{-\frac{\alpha(0)}{w_1\lambda_1}\int_0^t \beta(s)\mathrm{d}s}.$$

Remark 3.2. We remark from the first identity in (3.10) and assumption (3.6) that

$$w_1\lambda_1 - \alpha(t)\int_0^t \beta(s)\mathrm{d}s \geq w_1\lambda_1 - \alpha(0)\beta_1 > 0, \quad \text{for all } t \in \mathbb{R}^+.$$

4. Main result

In this section we establish an existence result for problem (1.1).

4.1. Local existence

Theorem 4.1. *Assume that* (3.6)-(3.11) *hold, given any* $(u_0, u_1) \in H_0^2(\Omega) \cap L^{p(\cdot)}(\Omega) \times L^2(\Omega)$. *Then problem* (1.1) *admits a solution* $u(t)$ *satisfying:*

$$u \in L^\infty(0, T; V \cap L^{p(\cdot)}(\Omega)), \tag{4.1}$$

where

$$V = \left\{\varphi \in H^2(\Omega) : \varphi = 0 \ \text{on}\ \Gamma\right\}.$$

Proof. Let w_j $(j = 1, 2, ...)$ satisfy the spectral problem

$$(w_j, v)_{H_0^2} = \lambda_j (w_j, v), \quad \forall v \in H_0^2,$$

where $(.,.)_{H_0^2}$ represents the inner product in H_0^2. The family of functions $\{w_1, w_2, ..., w_m\}$ yield a Galerkin basis for both H_0^2 and $L^2(\Omega)$.

For any $m \in \mathbb{N}$, let us put $V_m = Span\{w_1, w_2, ..., w_m\}$. We define

$$u_m(t) = \sum_{i=1}^m K_{jm}(t) w_j, \tag{4.2}$$

where K_{jm} satisfies:

$$
\begin{gathered}
(u_{ttm}(t), w_j) + w_1 (\Delta u_m, \Delta w_j) + w_2 (\nabla u_{mt}, \nabla w_j) \\
+a(u_m (t), w_j) + \left(|u_m|^{p(x)-2} u_m, w_j \right) \\
-\left(\alpha (t) \int_0^t \beta (t-s) \nabla u_m (s) \, \mathrm{d}s, \nabla w_j \right) + \lambda (g (u_{mt}), w_j) = b (f (u_m), w_j), \quad \text{(Pm)}
\end{gathered}
$$

$$
\begin{cases}
u_m(0) = u_{0m} = \sum_{i=1}^{m} \alpha_{im} w_j, \ u_{mt}(0) = u_{1m} = \sum_{i=1}^{m} \beta_{im} w_j \ , \\
\qquad u_{0m} \to u_0 \text{ in } V_m, \quad u_{1m} \to u_1 \text{ in } L^2(\Omega).
\end{cases} \tag{4.3}
$$

for $1 \leq j \leq m$, and

$$
a(\psi, \Psi) = \int_\Omega |\nabla \psi|^{m(x)-2} \nabla \psi \nabla \Psi \mathrm{d}x.
$$

As the family $\{w_1, w_2, ..., w_m\}$ is linearly independent, the problem (Pm) admits at least one local solution u_m in the interval $[0, t_m]$ verifying $u_m (t) \in L^2 (0, t_m; V_m)$ and $u_{mt} (t) \in L^2 (0, t_m; V_m)$. The estimate below will allow t_m to be independent of m.

A priori Estimate 1

Let us define

$$
(\beta o \nabla u) (t) = \int_0^t \beta (t-s) \int_\Omega |\nabla u (s) - \nabla u (t)|^2 \, \mathrm{d}x \mathrm{d}s,
$$

it is easy, by differentiating the term $\alpha (t) (\beta o \nabla u) (t)$ with respect to t, to show that

$$
\begin{aligned}
&\alpha (t) \int_\Omega \int_0^t \beta (t-s) \nabla u (s) \nabla u_t (t) \, \mathrm{d}x \mathrm{d}s \\
&= -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \alpha (t) (\beta o \nabla u) (t) - \alpha (t) |\nabla u (t)|^2 \int_0^t \beta (s) \, \mathrm{d}s \right\} \\
&\quad + \frac{1}{2} \alpha (t) (\beta' o \nabla u) (t) - \frac{1}{2} \alpha (t) \beta (t) |\nabla u (t)|^2 \\
&\quad + \frac{1}{2} \alpha' (t) (\beta o \nabla u) (t) - \frac{1}{2} \alpha' (t) |\nabla u (t)|^2 \int_0^t \beta (s) \, \mathrm{d}s.
\end{aligned} \tag{4.4}
$$

Next, replacing w_j in (Pm) by $u_{mt} (t)$, yields

$$
\begin{gathered}
(u_{ttm}(t), u_{mt}(t)) + a(u_m (t), u_{mt} (t)) + w_1 (\Delta u_m (t), \Delta u_{mt} (t)) \\
+ w_2 (\nabla u_m (t), \nabla u_{mt} (t)) \\
+ \left(|u_m|^{p(x)-2} u_m(t), u_{mt}(t) \right) - \alpha (t) \int_0^t \beta (t-s) (\nabla u_m (s), \nabla u_{mt} (t)) \, \mathrm{d}s \\
+ \lambda (g (u_{mt}), u_{mt} (t)) = b(f (u_m (t)), u_{mt}(t)).
\end{gathered} \tag{4.5}
$$

Using Young's inequality and (4.4), it results

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \frac{1}{2}\left|u_{mt}(t)\right|^2+\int_\Omega \frac{1}{m(x)}\left|\nabla u_m(t)\right|^{m(x)}\mathrm{d}x+\frac{1}{2}w_1\left|\Delta u_m\right|^2 \\ -\frac{1}{2}\left(\alpha(t)\int_0^t\beta(s)\,\mathrm{d}s\right)\left|\nabla u_m(t)\right|^2 \\ +\frac{1}{2}\alpha(t)\left(\beta o\nabla u_m\right)(t)+\int_\Omega\frac{1}{p(x)}\left|u_m(t)\right|^{p(x)}\mathrm{d}x-b\int_\Omega\widehat{f}\left(u_m(t)\right)\mathrm{d}x \end{pmatrix} \tag{4.6}$$

$$\begin{aligned} &+\lambda\left(g\left(u_{mt}\right),u_{mt}(t)\right)+w_2\left|\nabla u_{mt}(t)\right|^2 \\ &=\frac{1}{2}\alpha(t)\left(\beta' o\nabla u_m\right)(t)+\frac{1}{2}\alpha'(t)\left(\beta o\nabla u_m\right)(t) \\ &-\frac{1}{2}\left(\alpha(t)\beta(t)+\alpha'(t)\int_0^t\beta(s)\,\mathrm{d}s\right)\left|\nabla u_m(t)\right|^2. \end{aligned}$$

We denote by E_m the energy functional associated with problem (1.1):

$$\begin{aligned} E_m(t)=&\ \frac{1}{2}\left|u_{mt}(t)\right|^2+\frac{1}{2}w_1\left|\Delta u_m\right|^2-\frac{1}{2}\left(\alpha(t)\int_0^t\beta(s)\,\mathrm{d}s\right)\left|\nabla u_m(t)\right|^2 \\ &+\frac{1}{2}\alpha(t)\left(\beta o\nabla u_m\right)(t)+\int_\Omega\frac{1}{m(x)}\left|\nabla u_m(t)\right|^{m(x)}\mathrm{d}x \\ &+\int_\Omega\frac{1}{p(x)}\left|u_m(t)\right|^{p(x)}\mathrm{d}x-b\int_\Omega\widehat{f}\left(u_m(t)\right)\mathrm{d}x. \end{aligned} \tag{4.7}$$

Using the conditions (3.6), (3.10) and (3.11), we see that

$$\begin{aligned} E_m'(t)\le&\ \frac{1}{2}\alpha(t)\left(\beta' o\nabla u_m\right)(t)-\frac{1}{2}\left(\alpha(t)\beta(t)+\alpha'(t)\int_0^t\beta(s)\,\mathrm{d}s\right)\left|\nabla u_m(t))\right|^2 \\ &+\frac{1}{2}\alpha'(t)\left(\beta o\nabla u_m\right)(t)\le 0\quad\forall t\ge 0. \end{aligned} \tag{4.8}$$

The Young's inequality and (3.8), gives

$$\begin{aligned} -b\int_\Omega\widehat{f}\left(u_m(t)\right)\mathrm{d}x\ge&-\int_\Omega\frac{b}{p(x)}k_1(x)\left|u_m\right|\mathrm{d}x-\int_\Omega\frac{b}{p(x)}u_mf\left(x,u_m\right)\mathrm{d}x \qquad (4.9)\\ \ge&-\varepsilon_+\frac{1}{p_-^2}\int_\Omega\left|u_m(t)\right|^{p(x)}\mathrm{d}x-C_{\varepsilon_+}\int_\Omega\left|k_1(x)\right|^{p'(x)}\mathrm{d}x \\ &-\int_\Omega\frac{b}{p(x)}u_mf\left(x,u_m\right)\mathrm{d}x. \end{aligned}$$

Next, using hypothesis (3.9) and Young's inequality, we obtain

$$\int_{\Omega}\frac{b}{p(x)}u_m f(x,u_m)\,\mathrm{d}x \leq \int_{\Omega}\frac{b}{p(x)}\left|f(x,u_m)\right|\left|u_m\right|\mathrm{d}x$$

$$\leq \frac{l_1^2}{p_-}\varepsilon_+\int_{\Omega}\left(\left|u_m\right|^{2\theta}+\left|k_2(x)\right|^2\right)\mathrm{d}x+\frac{c(\varepsilon_+,p_-)}{p_-^2}\int_{\Omega}\left|u_m\right|^2\mathrm{d}x$$

$$\leq \frac{l_1^2}{p_-}\varepsilon_+\left(\int_{\Omega}\frac{p(x)-2\theta}{p(x)}\mathrm{d}x+2\theta\int_{\Omega}\frac{1}{p(x)}\left|u_m(t)\right|^{p(x)}\mathrm{d}x\right)+\frac{l_1^2}{p_-}\varepsilon_+\left\|k_2(x)\right\|_\infty^2$$

$$+C'(\varepsilon_+,p_-)+\frac{\varepsilon_+}{p_-^2}\int_{\Omega}\left|u_m(t)\right|^{p(x)}\mathrm{d}x \tag{4.10}$$

$$\leq \frac{l_1^2}{p_-}\varepsilon_+\left(\left|\Omega\right|\frac{p_+-2\theta}{p_-}+\frac{2\theta}{p_-}\int_{\Omega}\left|u_m(t)\right|^{p(x)}\mathrm{d}x\right)$$

$$+\frac{l_1^2}{p_-}\varepsilon_+\left\|k_2(x)\right\|_\infty^2+C'(\varepsilon_+,p_-)+\frac{\varepsilon_+}{p_-^2}\int_{\Omega}\left|u_m(t)\right|^{p(x)}\mathrm{d}x.$$

Now replace (4.10) in (4.9) and let $0<\varepsilon_+\leq\frac{p_-^2}{p_+\left(2+2\theta l_1^2\right)}$; by using (3.10), (3.12) and Remark 3.2 from (4.7), we obtain:

$$E_m(t)\geq\frac{1}{2}\left|u_{mt}(t)\right|^2+\frac{1}{2\lambda_1}\left(w_1\lambda_1-\alpha(0)\beta_1\right)\left|\Delta u_m(t)\right|^2 \tag{4.11}$$

$$+C_1\int_{\Omega}\left|\nabla u_m(t)\right|^{m(x)}\mathrm{d}x+C_2\int_{\Omega}\left|u_m(t)\right|^{p(x)}\mathrm{d}x-C_3\left(1+K_1+K_2\right),$$

or

$$\left|u_{mt}(t)\right|^2+\left|\Delta u_m(t)\right|^2+\int_{\Omega}\left|u_m(t)\right|^{p(x)}\mathrm{d}x+\int_{\Omega}\left|\nabla u_m(t)\right|^{m(x)}\mathrm{d}x$$

$$\leq C_4\left(E_m(t)+K_1+K_2+1\right), \tag{4.12}$$

where

$$C_1\geq\frac{1}{m_+},\ 0<C_2=\frac{p_-^2-p_+\left(2+2\theta l_1^2\right)\varepsilon_+}{p_-^2p_+},$$

$$C_3=\max\left(C_{\varepsilon_+};\frac{l_1^2}{p}\varepsilon_+;C'(\varepsilon_+,p_-)+\frac{l_1^2}{p_-}\varepsilon_+\frac{p_--2\theta}{p_-}\right),$$

$$C_4=\max\left(\frac{1}{\min\left(\frac{1}{2\lambda_1}\left(w_1\lambda_1-\alpha(0)\beta_1\right),C_1,C_2\right)},C_3\right).$$

Thus, it follows from (4.6), (4.8) and (4.12) that

$$\left|u_{mt}(t)\right|^2+\int_{\Omega}\left|\nabla u_m(t)\right|^{m(x)}\mathrm{d}x+\left|\Delta u_m\right|^2+\int_{\Omega}\left|u_m(t)\right|^{p(x)}\mathrm{d}x$$

$$+w_2\int_0^t\left|\nabla u_{mt}(s)\right|^2\mathrm{d}s+\lambda\int_0^t\left(g\left(u_{mt}(s)\right),u_{mt}(s)\right)\mathrm{d}s \tag{4.13}$$

$$\leq C_4\left(E_m(0)+K_1+K_2+1\right)\quad\text{for every } t\geq0$$

where $K_1 = \|k_1\|_\infty^2$, $K_2 = \|k_2\|_\infty^2$.

According to Hölder's inequality, using (3.8) and (3.9), we have

$$\left| b \int_\Omega \widehat{f}\,(u_m(0))\,\mathrm{d}x \right| \le \frac{b}{p_-} \int_\Omega |k_1(x)|\,|u_{0m}|\,\mathrm{d}x + \frac{b}{p_-} \int_\Omega |u_{0m}|\,|f(x, u_{0m})|\,\mathrm{d}x$$
$$\le C\left(|u_{0m}|^2 + \|k_1\|_\infty^2 + \int_\Omega |u_{0m}|^{p(x)}\,\mathrm{d}x + \|k_2\|_\infty^2 + |u_{0m}|^2 \right).$$

Therefore from (4.7) one has

$$E_m(0) = \frac{1}{2}|u_{1m}|^2 + \int_\Omega \frac{1}{m(x)} |\nabla u_{0m}|^{m(x)}\,\mathrm{d}x + \frac{1}{2}|\Delta u_{0m}|^2$$
$$+ \int_\Omega \frac{1}{p(x)} |u_{0m}|^{p(x)}\,\mathrm{d}x - b\int_\Omega \widehat{f}\,(u_{0m})\,\mathrm{d}x$$
$$\le C\left(|u_{1m}|^2 + \int_\Omega |\nabla u_{0m}|^{m(x)}\,\mathrm{d}x + |\Delta u_{0m}|^2 + \int_\Omega |u_{0m}|^{p(x)}\,\mathrm{d}x + |u_{0m}|^2 + K_1 + K_2 \right).$$

Then from (4.3) and (4.13), we obtain

$$|u_{mt}(t)|^2 + \int_\Omega |\nabla u_m(t)|^{m(x)}\,\mathrm{d}x + |\Delta u_m|^2 + \int_\Omega |u_m(t)|^{p(x)}\,\mathrm{d}x$$
$$+ w_2 \int_0^t |\nabla u_{mt}(s)|^2\,\mathrm{d}s + \lambda \int_0^t (g(u_{mt}(s)), u_{mt}(s))\,\mathrm{d}s \le C,$$

for some positive constant $C > 0$.

Gronwall's inequality and assumption (3.7) gives

$$\begin{cases} u_m \text{ is bounded in } L^\infty\left(0, T; H_0^2(\Omega) \cap L^{p(\cdot)}(\Omega)\right), \\ u_{mt} \text{ is bounded in } L^\infty\left(0, T; L^2(\Omega)\right), \\ g(u_{mt}).u_{mt} \text{ is bounded in } L^1(\Omega \times (0, T)), \\ u_{mt} \text{ is bounded in } L^2\left(0, T; L^{\sigma(\cdot)}(\Omega)\right), \\ \nabla u_{mt} \text{ is bounded in } L^2\left(0, T; L^2(\Omega)\right), \\ \nabla u_m \text{ is bounded in } L^\infty\left(0, T; L^{m(\cdot)}(\Omega)\right), \\ \Delta_{m(.)}(u_m) \text{ is bounded in } L^\infty\left(0, T; W^{-1, m'(\cdot)}(\Omega)\right). \end{cases} \tag{4.14}$$

Since $H_0^1 \hookrightarrow W_0^{1,p_+}(\Omega)$, we can use the standard projection arguments as in Lions [16]. Then from (Pm) and the estimates (4.14), we obtain

$$u_{ttm} \text{ is bounded in } L^2\left(0, T; H_0^{-1}(\Omega)\right). \tag{4.15}$$

To estimate the term $g(u_{mt}(t))$ we need the following lemma.

Lemma 4.2. *For all $m \in \mathbb{N}$ there exists $M > 0$ such that*

$$||g(u_{mt}(t))||_{L^{\frac{\sigma(x)}{\sigma(x)-1}}(Q)} \le M.$$

Proof. Thanks to Holder's, and Young's inequalities, from (3.7), we get

$$\begin{aligned}
\int_\Omega |g(u_{mt})|^{\frac{\sigma(x)}{\sigma(x)-1}}\,\mathrm{d}x &= \int_\Omega |g(u_{mt})|\,|g(u_{mt})|^{\frac{1}{\sigma(x)-1}}\,\mathrm{d}x \\
&\le \int_\Omega |g(u_{mt}(t))| \left(d_1 |u_{mt}(t)| + d_2 |u_{mt}(t)|^{\sigma(x)-1}\right)^{\frac{1}{\sigma(x)-1}} \mathrm{d}x \\
&\le C \int_\Omega |g(u_{mt}(t))| \left(|u_{mt}(t)|^{\frac{1}{\sigma(x)-1}} + |u_{mt}(t)|\right) \mathrm{d}x \\
&= C \int_\Omega |g(u_{mt}(t))|\,|u_{mt}(t)|^{\frac{1}{\sigma(x)-1}}\,\mathrm{d}x + C \int_\Omega |g(u_{mt}(t))|\,|u_{mt}(t)|\,\mathrm{d}x \\
&\le \frac{\sigma_+ - 1}{\sigma_+} \int_\Omega |g(u_{mt})|^{\frac{\sigma(x)}{\sigma(x)-1}}\,\mathrm{d}x + C(\sigma_+,\sigma_-) \int_\Omega |u_{mt}(t)|^{\frac{\sigma(x)}{\sigma(x)-1}}\,\mathrm{d}x \\
&\quad + C \int_\Omega |g(u_{mt}(t))|\,|u_{mt}(t)|\,\mathrm{d}x,
\end{aligned}$$

therefore

$$\begin{aligned}
\frac{1}{\sigma_+} \int_\Omega |g(u_{mt}(t))|^{\frac{\sigma(x)}{\sigma(x)-1}}\,\mathrm{d}x &\le C(\sigma_+,\sigma_-) \int_\Omega |u_{mt}(t)|^{\frac{\sigma(x)}{\sigma(x)-1}}\,\mathrm{d}x \\
+C \int_\Omega |g(u_{mt}(t))|\,|u_{mt}(t)|\,\mathrm{d}x &\le C\,\|u_{mt}(t)\|_2^{\frac{\sigma(x)}{\sigma(x)-1}} + C \int_\Omega |g(u_{mt}(t))|\,|u_{mt}(t)|\,\mathrm{d}x,
\end{aligned}$$

hence, estimates (4.14) gives

$$\int_0^T \int_\Omega |g(u_{mt}(t))|^{\frac{\sigma(x)}{\sigma(x)-1}}\,\mathrm{d}x\mathrm{d}t \le M. \tag{4.16}$$

□

By estimate (4.16)

$$g(u_{mt}(t)) \to g(u_t(t)) \ \ a.e. \text{ in } \Omega \times (0,T)$$

Therefore from Lions [16, Lemma 1.3] we infer that

$$g(u_{mt}) \to g(u_t) \ \text{ in } L^{\frac{\sigma(.)}{\sigma(.)-1}}(\Omega \times (0,T)) \ \text{ weak star.} \tag{4.17}$$

Passage to the limit

On the other hand, we have from (4.14)

$$\begin{cases}
u_m \longrightarrow u \text{ weak star in } L^\infty\left(0,T; H_0^2(\Omega) \cap L^{p(.)}(\Omega)\right), \\
\Delta^2 u_m \longrightarrow \Delta^2 u \text{ weak star in } L^\infty\left(0,T; H_0^2(\Omega) \cap L^{p(.)}(\Omega)\right), \\
u_{mt} \longrightarrow u_t \text{ weak star in } L^2\left(0,T; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right), \\
g(u_{mt}) \longrightarrow g(u_t) \text{ weak star in } L^{\frac{\sigma(.)}{\sigma(.)-1}}(\Omega \times (0,T)), \\
\Delta u_{mt}(t) \to \Delta u_t(t) \text{ weak star in } L^2\left(0,T; H^{-1}(\Omega)\right), \\
\Delta_{m(.)}(u_m) \to \psi \text{ weak star in } L^\infty\left(0,T; W^{-1,m'(.)}(\Omega)\right)
\end{cases} \tag{4.18}$$

By applying the Lions-Aubin compactness lemma, we obtain, for any $T>0$,

$$u_m \longrightarrow u \ \text{ strongly in } L^2\left(0,T;H_0^1(\Omega)\right). \tag{4.19}$$

Using the compactness of $H_0^1(\Omega)$ in $L^2(\Omega)$, it is easy to verify

$$\int_0^T\int_\Omega |u_m|^{p(.)-2}\, u_m v \mathrm{d}x\mathrm{d}t \to \int_0^T\int_\Omega |u|^{p(.)-2}\, uv\mathrm{d}x\mathrm{d}t \ \text{ for all } v\in L^{\sigma(.)}\left(0,T;H_0^1(\Omega)\right),$$

as $m\to\infty$.

Using growth conditions (3.9) and (4.18), we see that $\int_0^T\int_\Omega |f(u_m)|^{\frac{\theta+1}{\theta}}\,\mathrm{d}x\mathrm{d}t$ is bounded and

$$f(u_m) \longrightarrow f(u) \ \text{ a.e.in } \Omega\times(0,T),$$

then

$$f(u_m) \longrightarrow f(u) \ \text{ weak star in } L^{\frac{\theta+1}{\theta}}\left(0,T;L^{\frac{\theta+1}{\theta}}\right),$$

as $m\to\infty$, which implies that

$$\int_0^T\int_\Omega f(u_m)\, v\mathrm{d}x\mathrm{d}t \to \int_0^T\int_\Omega f(u)\, v\mathrm{d}x\mathrm{d}t \ \text{ for all } v\in L^{\theta+1}\left(0,T;H_0^1(\Omega)\right).$$

Passing to the limit in (Pm), we have

$$\begin{aligned}(u_{tt}(t),v)-(\psi,v)+w_1\left(\Delta^2u,v\right)-w_2\left(\Delta u_t,v\right)+\left(|u|^{p(.)-2}\,u,v\right) \qquad (4.20)\\ -\left(\alpha(t)\int_0^t\beta(t-s)\nabla u(s)\mathrm{d}s,\nabla v\right)+\lambda\left(g(u_t),v\right)=b\left(f(u),v\right) \quad \forall v\in W^{1,p(.)}(\Omega).\end{aligned}$$

Finally, by strong convergence, we can use a standard monotonicity argument as done in Lions [16] or Ma & Soriano [17] to show that $\psi=\Delta_{m(.)}(u)$. Then we infer that limit u satisfies (4.1) and

$$\begin{aligned}u_{tt}-\Delta_{m(.)}(u)+w_1\Delta^2u-w_2\Delta u_t+\alpha(t)\int_0^t\beta(t-s)\,\Delta u(s))\mathrm{d}s+|u|^{p(.)-2}\,u\\ +\lambda g(u_t)=bf(u).\end{aligned}$$

From where the proof of theorem (4.1). □

4.2. Uniqueness

In this subsection, the uniqueness of the solution will be proven.

Theorem 4.3. *Let the assumptions of theorem 4.1 hold. Assume further that*

$$p_+\le\frac{2n-2}{n-2},\ n\ne2\ \ (p_+<\infty \ \textit{if}\ n\le2) \tag{4.21}$$

$$m_+\le\frac{2n-2}{n-2},\ n\ne2\ \ (m_+<\infty \ \textit{if}\ n\le2), \tag{4.22}$$

$$1<\theta\le\frac{p_-}{2}, \tag{4.23}$$

Then, there exists a unique solution u to problem 1.1 and it satisfies (4.1).

Proof. Let u, v be two weak solutions of problem 1.1, and set $\Psi = u - v$. Then, Ψ satisfies the equation

$$\begin{aligned} &\Psi_{tt}(t) - \left(\Delta_{m(.)}u(t) - \Delta_{m(.)}v(t)\right) + w_1\Delta^2\Psi(t) - w_2\Delta\Psi'(t) \\ &+\lambda\left(g\left(u_t(t)\right) - g\left(v_t(t)\right)\right) + \left(|u(t)|^{p(.)-2}u(t) - |v(t)|^{p(.)-2}v(t)\right) \\ &+\alpha(t)\int_0^t \beta(t-s)\,\Delta\Psi(s)\mathrm{d}s = b\left(f\left(u(t)\right) - f\left(v(t)\right)\right) \end{aligned} \tag{4.24}$$

in $L^2\left(0,T;L^2(\Omega)\right)$, $T>0$, with boundary conditions and null initial data.
As $\Psi' \in L^2\left(0,T;H_0^1(\Omega)\right)$, multiplying above equation by $\Psi'(t)$, to get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Psi_t(t)|^2 + w_1\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\Psi(t)|^2 + w_2|\nabla\Psi_t|^2 + \left(g(u_t) - g(v_t), u_t - v_t\right) \\ &+\left(|\nabla u|^{m(.)-2}\nabla u - |\nabla v|^{m(.)-2}\nabla v, \nabla\Psi_t\right) = \int_\Omega \left(|v|^{p(.)-2}v - |u|^{p(.)-2}u\right)\Psi_t\mathrm{d}x \\ &+\left(f(u) - f(v), \Psi_t\right) + \alpha(t)\int_\Omega\int_0^t \beta(t-s)\nabla\Psi(s)\nabla\Psi_t(t)\,\mathrm{d}s\mathrm{d}x. \end{aligned} \tag{4.25}$$

From (3.7) we have:

$$\left(g(u_t) - g(v_t), u_t - v_t\right) \geq 0.$$

Thanks to Hölder's inequality, we estimated the first term on the right hand side of (4.25) as follows:

$$\begin{aligned} \left|\int_\Omega (|v|^{p(x)-2}v - |u|^{p(x)-2}u)\Psi_t\mathrm{d}x\right| &\leq (p_+ - 1)\int_\Omega \sup\left(|u|^{p(x)-2}, |v|^{p(x)-2}\right)|\Psi||\Psi_t|\,\mathrm{d}x \\ &\leq (p_+ - 1)\int_\Omega \left(|u|^{p_+-2} + |v|^{p_+-2} + |u|^{p_--2} + |v|^{p_--2}\right)|\Psi||\Psi_t|\,\mathrm{d}x \\ &\leq C\begin{pmatrix} \|u\|^{p_+-2}_{L^{n(p_+-2)}(\Omega)} + \|v\|^{p_+-2}_{L^{n(p_+-2)}(\Omega)} \\ + \|u\|^{p_--2}_{L^{n(p_--2)}(\Omega)} + \|v\|^{p_--2}_{L^{n(p_--2)}(\Omega)} \end{pmatrix} ||\Psi(t)||_{L^q(\Omega)}|\Psi_t(t)|, \end{aligned}$$

where $\frac{1}{n} + \frac{1}{q} + \frac{1}{2} = 1$, and from (4.21), $n(p_- - 2) \leq n(p_+ - 2) \leq \frac{2n}{n-2} = q$ which gives by estimate (4.1), Young's inequality and as $H_0^1(\Omega) \subset L^q(\Omega)$, that:

$$\begin{aligned} &\left|\int_\Omega (|v|^{p(x)-2}v - |u|^{p(x)-2}u)\Psi_t\mathrm{d}x\right| \\ &\leq C\begin{pmatrix} \|\nabla u\|^{p_+-2}_{L^2(\Omega)} + \|\nabla v\|^{p_+-2}_{L^2(\Omega)} \\ + \|\nabla u\|^{p_--2}_{L^2(\Omega)} + \|\nabla v\|^{p_--2}_{L^2(\Omega)} \end{pmatrix} ||\nabla\Psi(t)||_{L^2(\Omega)}|\Psi_t(t)| \\ &\leq C\left(|\nabla\Psi(t)|^2 + |\Psi_t(t)|^2\right). \end{aligned}$$

By the same manner and by condition (4.21), we have

$$\begin{aligned}
&\left|\int_{\Omega}(|\nabla u|^{m(x)-2}\nabla u-|\nabla v|^{m(x)-2}\nabla v\nabla\Psi_t\mathrm{d}x\right| \\
&\leq (m_+-1)\int_{\Omega}\sup\left(|\nabla u|^{m(x)-2},|\nabla v|^{m(x)-2}\right)|\nabla\Psi|\,|\nabla\Psi_t|\,\mathrm{d}x \\
&\leq C\begin{pmatrix} \|u\|^{m_+-2}_{L^{n(m_+-2)}(\Omega)}+\|v\|^{m_+-2}_{L^{n(m_+-2)}(\Omega)} \\ +\|u\|^{m_--2}_{L^{n(m_--2)}(\Omega)}+\|v\|^{m_--2}_{L^{n(m_--2)}(\Omega)}\end{pmatrix}||\Psi(t)||_{L^q(\Omega)}\,|\Psi'(t)|, \\
&\leq C\begin{pmatrix} \|\nabla u\|^{m_+-2}_{L^2(\Omega)}+\|\nabla v\|^{m_+-2}_{L^2(\Omega)} \\ +\|\nabla u\|^{p_--2}_{L^2(\Omega)}+\|\nabla v\|^{p_--2}_{L^2(\Omega)}\end{pmatrix}||\nabla\Psi(t)||_{L^2(\Omega)}\,|\Psi_t(t)| \\
&\leq C\left(|\nabla\Psi(t)|^2+|\Psi_t(t)|^2\right).
\end{aligned}$$

Now setting $U_\zeta=\zeta u+(1-\zeta)v$, $0\leq\zeta\leq1$, from the growth condition it follows that

$$\begin{aligned}
\left|\int_0^t\int_{\Omega}|f(u)-f(v)|\,|\Psi_t|\,\mathrm{d}x\mathrm{d}t\right| &= \left|\int_0^t\int_{\Omega}\int_0^1\frac{\mathrm{d}}{\mathrm{d}\zeta}f(U_\zeta)\,\mathrm{d}\zeta\Psi_t\mathrm{d}x\mathrm{d}t\right| \\
&\leq \int_0^t\int_{\Omega}\left|\int_0^1\frac{\mathrm{d}}{\mathrm{d}\zeta}f(U_\zeta)\,\mathrm{d}\varepsilon\right|\,|\Psi_t|\,\mathrm{d}x\mathrm{d}s \\
&\leq \int_0^t\int_{\Omega}\int_0^1\left|\frac{\mathrm{d}}{\mathrm{d}\zeta}f(U_\zeta)\,\mathrm{d}\zeta\right|\,|\Psi_t|\,\mathrm{d}x\mathrm{d}s \\
&\leq l_1\int_0^t\int_{\Omega}\int_0^1\left(|U_\zeta|^{\theta-1}+|k_3(x)|\right)|u-v|\,|\Psi_t|\,\mathrm{d}\zeta\mathrm{d}x\mathrm{d}s \\
&\leq C\int_0^t\int_{\Omega}\left(|u|^{\theta-1}+|v|^{\theta-1}+|k_3(x)|\right)|\Psi(s)|\,|\Psi_t(s)|\,\mathrm{d}x\mathrm{d}s=I.
\end{aligned}$$

Using generalized Hölder's, Young's inequalities, estimates (4.1), and let λ satisfy:

$$1<\lambda+1\leq\min\left(\frac{n}{(n-2)(\theta-1)},\frac{n}{n-2}\right),\ n\neq2\ \ (\lambda<\infty \text{ if } n\leq2) \tag{4.26}$$

from (4.23), the following estimates hold,

$$\begin{aligned}
I &\leq C\int_0^t\left\|l_1\left(|u|^{\theta-1}+|v|^{\theta-1}+|k_3(x)|\right)\right\|^{\lambda}_{2(\lambda+1)}||\Psi||_{2(\lambda+1)}\,\|\Psi_t\|_2 \\
&\leq C\int_0^t\left(\left\||u|^{\theta-1}\right\|^{\lambda}_{2(\lambda+1)}+\left\||v|^{\theta-1}\right\|^{\lambda}_{2(\lambda+1)}+\|k_3(x)\|^{\lambda}_{2(\lambda+1)}\right)||\Psi||_{2(\lambda+1)}\,\|\Psi_t\|_2\,\mathrm{d}s \\
&\leq C\int_0^t\left(\|\nabla u\|_2^{\lambda(\theta-1)}+\|\nabla v\|_2^{\lambda(\theta-1)}+\|k_3(x)\|^{\lambda}_{\infty}\right)||\nabla\Psi||_2\,\|\Psi_t\|_2\,\mathrm{d}s \\
&\leq C\int_0^t\left(|\Psi_t(s)|^2+|\nabla\Psi(s)|^2\right)\mathrm{d}s.
\end{aligned}$$

because by (4.26) we have $||\Psi||_{2(\lambda+1)}\leq||\nabla\Psi||_2$.

Combining the above inequalities with identity (4.4), from (4.25), we derive

$$\begin{aligned}
&\frac{1}{2}\left|\Psi_t(t)\right|^2+\frac{1}{2}C\left(w_1\lambda_1-\alpha(t)\int_0^t\beta(s)\,\mathrm{d}s\right)\left|\nabla\Psi(t)\right|^2\\
&+C_2\int_0^t\left|\nabla\Psi_t(s)\right|^2\mathrm{d}s+\frac{1}{2}\alpha(t)\left(\beta o\nabla\Psi\right)(t)\\
&\leq C\int_0^t\left(\left|\Psi_t(s)\right|^2+\left|\nabla\Psi(s)\right|^2\right)\mathrm{d}s+\frac{1}{2}\int_0^t\alpha'(s)\left(\beta o\nabla\Psi\right)(s)\,\mathrm{d}s\\
&+\frac{1}{2}\int_0^t\alpha(s)\left(\beta' o\nabla\Psi\right)(s)\,\mathrm{d}s-\frac{1}{2}\int_0^t\left(\alpha(s)\beta(s)+\alpha'(s)\int_0^s\beta(\zeta)\,\mathrm{d}\zeta\right)\left|\nabla\Psi(s)\right|^2\mathrm{d}s
\end{aligned}$$

Then, from remark (3.2), assumptions (3.10) gives

$$\left|\Psi_t(t)\right|^2+\left(w_1\lambda_1-\alpha(0)\beta_1\right)\left|\nabla\Psi(t)\right|^2\leq C\int_0^t\left(\left|\Psi_t(s)\right|^2+\left|\nabla\Psi(s)\right|^2\right)\mathrm{d}s.$$

and then by Gronwall's inequality we deduce that: $\Psi(t)=\Psi(0)=0$ in $H_0^2(\Omega)$. □

To study the global existence of the energy function, we define some functionals and establish several lemmas. Let the functions:

$$\begin{aligned}
I(t)=I(u(t))=&\frac{p(x)}{4}\left(w_1\lambda_1-\alpha(t)\int_0^t\beta(s)\,\mathrm{d}s\right)\left|\nabla u(t)\right|^2\\
&-b\int_\Omega f(u(t))\,u(t)\,\mathrm{d}x-b\int_\Omega k_1(x)\left|u(t)\right|\mathrm{d}x;
\end{aligned}\tag{4.27}$$

$$J(t)=J(u(t))=\frac{1}{2}\left(\lambda_1 w_1-\alpha(t)\int_0^t\beta(s)\,\mathrm{d}s\right)\left|\nabla u(t)\right|^2-b\int_\Omega\widehat{f}(x,u)\,\mathrm{d}x;\tag{4.28}$$

$$\begin{aligned}
E(t)=E(u(t),u_t(t))\geq{}&J(u(t))+\frac{1}{2}\left|u_t(t)\right|^2+\int_\Omega\frac{1}{p(x)}\left|u(t)\right|^{p(x)}\mathrm{d}x\\
&+\int_\Omega\frac{1}{m(x)}\left|\nabla u(t)\right|^{m(x)}\mathrm{d}x+\frac{1}{2}\alpha(t)\left(\beta o\nabla u\right)(t).
\end{aligned}\tag{4.29}$$

And the set as

$$W=\left\{u:u\in H_0^2(\Omega),\ I(t)>0\right\}\cup\{0\}.\tag{4.30}$$

where

$$\begin{aligned}
E(t)={}&\frac{1}{2}\left|u_t(t)\right|^2+\frac{1}{2}w_1\left|\Delta u\right|^2-\frac{1}{2}\left(\alpha(t)\int_0^t\beta(s)\,\mathrm{d}s\right)\left|\nabla u(t)\right|^2+\frac{1}{2}\alpha(t)\left(\beta o\nabla u\right)(t)\\
&+\int_\Omega\frac{1}{m(x)}\left|\nabla u(t)\right|^{m(x)}\mathrm{d}x+\int_\Omega\frac{1}{p(x)}\left|u(t)\right|^{p(x)}\mathrm{d}x-b\int_\Omega\widehat{f}(u(t))\,\mathrm{d}x.
\end{aligned}\tag{4.31}$$

5. Global existence

In this section we show that the solution of problem 1.1 global in in infinite time under the assumption

$$E(0) < 4(w_1\lambda_1 - \alpha(0)\beta_1)\left(\frac{p_-(\lambda_1 w_1 - \alpha(0)\beta_1)}{4\left(l_1 + l_1 \|k_2(x)\|_\infty + \|k_1(x)\|_\infty\right) bC_*^{\theta+1}}\right)^{\frac{2}{\theta-1}}.$$

and

$$p_+ \le \frac{2n}{n-2},\ n \ne 2\ \ (p_+ < \infty \text{ if } n \le 2).$$

The next lemma shows that our energy functional (4.29) is a nonincreasing function along the solution of (1.1).

Lemma 5.1. *$E(t)$ is a nonincreasing for $t \ge 0$ and*

$$E'(t) = -w_2 |\nabla u_t|^2 - \lambda \int_\Omega u_t(t)\, g(u_t(t))\,\mathrm{d}x + \frac{1}{2}\alpha'(t)\int_\Omega (\beta o \nabla u)(t)\,\mathrm{d}x$$
$$+\frac{1}{2}\alpha(t)\int_\Omega (\beta' o \nabla u)(t)\,\mathrm{d}x - \frac{1}{2}\left(\alpha(t)\beta(t) + \alpha'(t)\int_0^t \beta(s)\,\mathrm{d}s\right)|\nabla u(t)|^2 \le 0. \quad (5.1)$$

Proof. Multiplying the equation of (1.1) by u_t and integrating by parts over Ω, using (3.6), (3.7), (3.10) and remark 3.2, summing up the product results, obtains

$$E(t) - E(0) = -w_2 \int_0^t |\nabla u_t(s)|^2\,\mathrm{d}s - \lambda \int_0^t \int_\Omega u_t(s)\, g(u_t(s))\,\mathrm{d}x\mathrm{d}s$$
$$+\frac{1}{2}\int_0^t \alpha'(t)\int_\Omega (\beta o \nabla u)(s)\,\mathrm{d}x\mathrm{d}s + \frac{1}{2}\int_0^t \alpha(s)\int_\Omega (\beta' o \nabla u)(t)\,\mathrm{d}x\mathrm{d}s$$
$$-\frac{1}{2}\int_0^t \left(\alpha(s)\beta(s) + \alpha'(s)\int_0^s \beta(\zeta)\,\mathrm{d}\zeta\right)|\nabla u(s)|^2\,\mathrm{d}s \le 0 \ \text{ for } t \ge 0. \qquad \square$$

Lemma 5.2. *Let (3.6) and (3.8) hold, suppose $u_0 \in W$ and $u_1 \in H_0^1(\Omega)$ such that*

$$\gamma = bC_*^{\theta+1}\left(4\frac{E(0)}{w_1\lambda_1 - \alpha(0)\beta_1}\right)^{\frac{\theta-1}{2}}\left(l_1 + l_1\|k_2(x)\|_\infty + \|k_1(x)\|_\infty\right) \quad (5.2)$$
$$< \frac{p_-}{4}(\lambda_1 w_1 - \alpha(0)\beta_1).$$

then $u \in W$ for each $t \ge 0$, where C_* is the best Poincar's, Sovolev constant depending only on $p(x)$ and on Ω, which satisfy $2 < p(x) \le p_+ \le \frac{2n}{n-2}\,(n \ge 3)$ $(2 \le p_+ < \infty$ if $n = 1, 2)$.

$$\|u(t)\|_{p(x)} \le C_* \|\nabla u(t)\|_2 \quad \forall u \in H_0^1(\Omega).$$

Proof. Since $I(0) > 0$, by the continuity, there exists $0 < T_m < T$ such

$$I(t) \ge 0 \ \text{ in } [0, T_m],$$

this gives from (4.28) and (3.8):

$$\begin{aligned} E\left(t\right) \geq J\left(t\right) &= \frac{1}{p\left(x\right)} I\left(t\right) + \frac{1}{4}\left(\lambda_1 w_1 - \alpha\left(t\right)\int_0^t \beta\left(s\right)\mathrm{d}s\right)\left|\nabla u\right|^2 \\ &+ \frac{b}{p\left(x\right)}\left(\int_\Omega f\left(u\right)u\mathrm{d}x + \int_\Omega k_1\left(x\right)\left|u\right|\mathrm{d}x - p\left(x\right)\int_\Omega \widehat{f}\left(x\right)\mathrm{d}x\right) \\ &\geq \frac{1}{4}\left(\lambda_1 w_1 - \alpha\left(t\right)\int_0^t \beta\left(s\right)\mathrm{d}s\right)\left|\nabla u\right|^2. \end{aligned} \tag{5.3}$$

since by (3.8) we have

$$\int_\Omega f\left(u\right)u\mathrm{d}x + \int_\Omega k_1\left(x\right)\left|u\right|\mathrm{d}x - p\left(x\right)\int_\Omega \widehat{f}\left(x\right)\mathrm{d}x \geq 0$$

Then by using (5.3), (4.29), (5.1) and remark 3.2, we obtain

$$\begin{aligned} \left|\nabla u\right|^2 &\leq 4\left(\lambda_1 w_1 - \alpha\left(t\right)\int_0^t \beta\left(s\right)\mathrm{d}s\right)^{-1} E\left(t\right) \\ &\leq 4\left(\lambda_1 w_1 - \alpha\left(t\right)\int_0^t \beta\left(s\right)\mathrm{d}s\right)^{-1} E\left(0\right). \end{aligned} \tag{5.4}$$

By recalling (3.9), Sobolev-Poincaré's embedding ($\theta + 1 \leq p$), condition (5.2), estimate (5.4) and Cauchy-Schwartz's inequality, we have the following estimates:

$$\begin{aligned} & b\int_\Omega f\left(u\right)u\mathrm{d}x + b\int_\Omega k_1\left(x\right)\left|u\right|\mathrm{d}x \leq b\int_\Omega \left|f\left(u\right)\right|\left|u\right|\mathrm{d}x + b\int_\Omega \left|k_1\left(x\right)\right|\left|u\right|\mathrm{d}x \\ &\leq bl_1\int_\Omega \left|u\right|^{\theta+1}\mathrm{d}x + bl_1\int_\Omega \left|k_2\left(x\right)\right|\left|u\right|\mathrm{d}x + b\int_\Omega \left|k_1\left(x\right)\right|\left|u\right|\mathrm{d}x \\ &\leq bl_1\left\|u\left(t\right)\right\|_{\theta+1}^{\theta+1} + b\left(l_1\left|\left|k_2\left(x\right)\right|\right|_\infty + \left|\left|k_1\left(x\right)\right|\right|_\infty\right)\left\|u\left(t\right)\right\|_{\theta+1}^{\theta+1} \\ &\leq bl_1 C_*^{\theta+1}\left|\nabla u(t)\right|^{\theta+1} \\ &+ bC_*^{\theta+1}\left(l_1\left|\left|k_2\left(x\right)\right|\right|_\infty + \left|\left|k_1\left(x\right)\right|\right|_\infty\right)\left|\nabla u(t)\right|^{\theta+1} \\ &= bl_1 C_*^{\theta+1}\left|\nabla u(t)\right|^{\theta-1}\left|\nabla u(t)\right|^2 \\ &+ bC_*^{\theta+1}\left(l_1\left|\left|k_2\left(x\right)\right|\right|_\infty + \left|\left|k_1\left(x\right)\right|\right|_\infty\right)\left|\nabla u(t)\right|^{\theta-1}\left|\nabla u(t)\right|^2 \\ &\leq bC_*^{\theta+1}\left(4\left(\lambda_1 w_1 - \alpha\left(t\right)\int_0^t \beta\left(s\right)\mathrm{d}s\right)^{-1} E\left(0\right)\right)^{\frac{\theta-1}{2}} \\ &\times\left(l_1 + l_1\left|\left|k_2\left(x\right)\right|\right|_\infty + \left|\left|k_1\left(x\right)\right|\right|_\infty\right)\left|\nabla u\right|^2 \\ &\leq bC_*^{\theta+1}\left(4\frac{E\left(0\right)}{w_1\lambda_1 - \alpha\left(0\right)\beta_1}\right)^{\frac{\theta-1}{2}} \times\left(l_1 + l_1\left|\left|k_2\left(x\right)\right|\right|_\infty + \left|\left|k_1\left(x\right)\right|\right|_\infty\right)\left|\nabla u\right|^2 \\ &< \frac{p_-}{4}\left(\lambda_1 w_1 - \alpha\left(0\right)\beta_1\right)\left|\nabla u\right|^2 \\ &\leq \frac{p\left(x\right)}{4}\left(\lambda_1 w_1 - \alpha\left(t\right)\int_0^t \beta\left(s\right)\mathrm{d}s\right)\left|\nabla u\right|^2 \text{ on } \left[0, T_m\right]. \end{aligned} \tag{5.5}$$

Therefore, from (4.27), we conclude that $I(t) > 0$ for all $t \in [0, T_m]$. By repeating this procedure, and using the fact that

$$\lim_{t\to T_m} bC_*^{\theta+1}\left(4\frac{E(t)}{w_1\lambda_1-\alpha(0)\beta_1}\right)^{\frac{\theta-1}{2}}\left(l_1+l_1\|k_2(x)\|_\infty+\|k_1(x)\|_\infty\right)\le D$$
$$<\frac{p_-}{4}\left(\lambda_1 w_1-\alpha(0)\beta_1\right).$$

T_m is extended to T. $\square$

Theorem 5.3. *Let the assumptions of theorem 4.1 hold. Let $u_0 \in W$ satisfying* (5.2)*. Then, the solution gotten in of theorem 4.1 is global.*

Proof. It sufficient independently to t to show that

$$|u_t|^2+|\nabla u|^2+\int_\Omega|\nabla u(t)|^{m(x)}\,\mathrm{d}x+\int_\Omega|u(t)|^{p(x)}\,\mathrm{d}x$$

is bounded.

For this aim, we use (4.27), (4.29), (3.8), (3.10) and Lemma 5.2 to obtain:

$$\begin{aligned}
E(0)\ge E(t)\ge{}&\frac{1}{2}\left(\lambda_1w_1-\alpha(t)\int_0^t\beta(s)\,\mathrm{d}s\right)|\nabla u(t))|^2-b\int_\Omega\widehat{f}(x,u)\,\mathrm{d}x\\
&+\frac{1}{2}|u_t(t)|^2+\int_\Omega\frac{1}{m(x)}|\nabla u(t)|^{m(x)}\,\mathrm{d}x+\int_\Omega\frac{1}{p(x)}|u(t)|^{p(x)}\,\mathrm{d}x+\frac{1}{2}\alpha(t)(\beta o\nabla u)(t)\\
\ge{}&\frac{1}{4}\left(\lambda_1w_1-\alpha(t)\int_0^t\beta(s)\,\mathrm{d}s\right)|\nabla u(t))|^2+\frac{1}{p(x)}I(t)\\
&+\frac{b}{p(x)}\left(\int_\Omega f(u)\,u\mathrm{d}x+\int_\Omega k_1(x)|u|\,\mathrm{d}x-p(x)\int_\Omega\widehat{f}(x,u)\,\mathrm{d}x\right)\\
&+\frac{1}{2}|u_t(t)|^2+\int_\Omega\frac{1}{m(x)}|\nabla u(t)|^{m(x)}\,\mathrm{d}x+\int_\Omega\frac{1}{p(x)}|u(t)|^{p(x)}\,\mathrm{d}x\\
\ge{}&\frac{1}{4}\left(\lambda_1w_1-\alpha(t)\int_0^t\beta(s)\,\mathrm{d}s\right)|\nabla u(t)|^2\\
&+\frac{1}{2}|u_t(t)|^2+\int_\Omega\frac{1}{m(x)}|\nabla u(t)|^{m(x)}\,\mathrm{d}x+\int_\Omega\frac{1}{p(x)}|u(t)|^{p(x)}\,\mathrm{d}x\\
\ge{}&\frac{1}{4}\left(\lambda_1w_1-\alpha(0)\beta_1\right)|\nabla u(t))|^2+\frac{1}{2}|u_t(t)|^2\\
&+\int_\Omega\frac{1}{m(x)}|\nabla u(t)|^{m(x)}\,\mathrm{d}x+\int_\Omega\frac{1}{p(x)}|u(t)|^{p(x)}\,\mathrm{d}x.
\end{aligned}$$

Therefore

$$|u_t(t)|^2+|\nabla u(t)|^2+\int_\Omega|\nabla u(t)|^{m(x)}\,\mathrm{d}x+\int_\Omega|u(t)|^{p(x)}\,\mathrm{d}x$$
$$\le\max\left(p^+,m^+,4\left(\lambda_1w_1-\alpha(0)\beta_1\right)^{-1}\right)E(0),$$

These estimates ensure that the solution $u(t)$ exist globally in $[0,+\infty[$. $\square$

Example 5.4. Consider the following functions:

$$f(x,u) = a(x)|u|^{\varpi-2}u - b(x)|u|^{\gamma-2}u$$

with appropriate functions $a(x)$ and $b(x)$, where $\varpi > \gamma \geq 1$.

$$\begin{array}{rcl} g(u_t(t)) & = & |u_t(t)|^{\sigma(x)-2}u_t(t); \quad \sigma(x) \text{ satisfies conditions in } (3.7); \\ \Delta_{m(x)}u & = & div\left(|\nabla u|^{m-2}\nabla u\right); \quad m(x) = m > 2. \end{array}$$

Then, problem (1.1), is reduced to the following problem

$$\begin{cases} u_{tt} - div\left(|\nabla u|^{m-2}\nabla u\right) + w_1\Delta^2 u(t) - w_2\Delta u_t(t) + \alpha(t)\displaystyle\int_0^t \beta(t-s)\Delta u(s)\,\mathrm{d}s \\ \qquad +\lambda|u_t(t)|^{\sigma(x)-2}u_t(t) + |u|^{p(x)-2}u(t) = bf(u(t)) \text{ in } \Omega\times\mathbb{R}^+, \\ \qquad u = \partial_\eta u = 0 \text{ on } \Gamma\times[0,+\infty[, \\ \qquad u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \text{ in } \Omega, \end{cases} \tag{P}$$

Since f, g satisfies hypotheses (3.7)-(3.9). Then, Theorems (4.1), (4.3) and (5.3) are verified for problem (P), which gives importance to this general problem.

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A two-steps fixed-point method for the simplicial cone constrained convex quadratic optimization

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Abstract. In this paper, we deal with the resolution of the simplicial cone constrained convex quadratic optimization (abbreviated SCQO). It is known that the optimality conditions of SCQO is only a standard linear complementarity problem (LCP). Under a suitable condition, the solution of LCP is equivalent to find the solution of an absolute value equations AVE. For its numerical solution, we propose an efficient two-steps fixed point iterative method for solving the AVE. Moreover, we show that this method converges globally linear to the unique solution of the AVE and which is in turn an optimal solution of SCQO. Some numerical results are reported to demonstrate the efficiency of the proposed algorithm.

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1. Introduction

Consider the simplicial cone constrained convex quadratic optimization SCQO:

$$\min_x \left[f(x) = \frac{1}{2} x^T Q x + x^T b + c \right] \text{ s.t. } x \in \mathbb{S} \tag{1.1}$$

where $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, $b \in \mathbb{R}^n, c \in \mathbb{R}^n$, and

$$\mathbb{S} = \left\{ Ax \mid x \in \mathbb{R}^n_+ \right\}$$

is the simplicial cone associated with the nonsingular matrix $A \in \mathbb{R}^{n\times n}$. The importance of quadratic programming lies in its theoretical properties, its applications in

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different scientific fields and several disciplines such as economics, finance, telecommunications, medicine and the engineering sciences. Another great advantage of the quadratic case is that we can transform several real and academic problems (polynomial minimization problems, least squares problems in numerical analysis, etc.) into an equivalent quadratic problem without loss of generality. Simplicial cone constrained convex quadratic programming is equivalent to the problem of projecting the point onto a simplicial cone (see [5, 6, 9]), with its KKT optimality conditions consisting a linear complementarity problem (see [8, 19]). From this optimality conditions, under suitable conditions, the convex quadratic programming under a simplicial cone constraints is equivalent to finding the unique solution of the following absolute value equation:

$$(A^TQA + I)x + (A^TQA - I)|x| = -A^Tb. \tag{1.2}$$

This equation is a special case of the general absolute value equations AVE of the type:

$$\bar{A}x - \bar{B}|x| = \bar{b}$$

where $\bar{A}, \bar{B}$ are given $(n \times n)$ real square matrices and $\bar{b} \in \mathbb{R}^n$. The AVE was first introduced by Rohn [18] and investigated in more general context in Mangasarian (see [16]). Other studies for the AVE can be found in [1, 3, 2, 7, 10, 12, 13, 15, 17]. Besides some numerical methods are used to solve it. In particular, Mangasarian in [14] proposed a semi-smooth Newton's method for solving the AVE, and under suitable conditions he showed the finite and linear convergence to a solution of the AVE. However, other numerical approaches focus on reformulating the AVE as an horizontal linear complementarity problems (HLCP) (see [4]), where they introduce an infeasible path-following interior-point method for solving the AVE by using is equivalent reformulations as an HLCP. In this paper, we propose a new two-steps fixed point iterative method for solving the AVE (1.2) which is introduced in [11], and under a new mild assumption we show that this method is always well-defined and the generated sequence converges globally and linearly to the unique solution of the AVE from any starting initial point. Finally, numerical results are provided to illustrate the efficiency of this algorithm to solving the SCQO.

Our paper is organized as follows. In section 2, some notations and basic results used in the paper are presented. In section 3, the reformulation of problem (1.1) as an absolute value equation AVE and the unique solvability of AVE is studied. Any solution of the AVE generates a solution of our convex quadratic programming problem SCQO. In section 4, a description and a convergence property of the two-steps fixed point iterative method for solving the AVE are stated. In section 5, numerical results are presented. We end this paper with a conclusion and some remarks in section 6.

2. Preliminaries

Let $\mathbb{R}^n$ be the Euclidean space provided with the usual scalar product $\langle x, y\rangle = x^Ty$ where x and y are two vectors of $\mathbb{R}^n$ and x^T is the transpose of x. The nonegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$. For $x \in \mathbb{R}^n_+$, we write $x \geq 0$, and means that $x_i \geq 0$, $\forall i$. For $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, we denote by:

$$x_i^+ := \max(0, x_i),\ x_i^- := \max(0, -x_i),\ |x| := (|x_1|, \ldots, |x_n|)^T.$$

It is easy from the definitions of x^+ and x^-, to conclude that:

$$x = x^+ - x^-,\ x^+ \in \mathbb{R}^n_+,\ x^- \in \mathbb{R}^n_+, \langle x^+, x^- \rangle = 0, |x| = x^+ + x^-, \forall x \in \mathbb{R}^n.$$

For $x \in \mathbb{R}^n$, sign(x) denotes a vector with components equal to 1, 0, -1 depending on whether the corresponding component of x is positive, zero or negative. We denote by $\mathbb{R}^{n\times n}$ the vector space of real square matrices of order n, the identity matrix is denoted by I_n. If $x \in \mathbb{R}^n$ then $X = Diag(x)$ denotes the $n \times n$ diagonal matrix with $X_{ii} = x_i$, $\forall i = 1, \ldots, n$. Let $A \in \mathbb{R}^{n\times n}$, its spectral matrix norm is denoted by $\|A\| := \max\{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\}$, where $\|x\|$ denotes the Euclidean norm, this definition implies:

$$\|Ax\| \leq \|A\| \|x\|,\ \|AB\| \leq \|A\| \|B\|,\ \forall A, B \in \mathbb{R}^{n\times n}.$$

For a matrix M, $\rho(M)$ denote its spectral radius. In addition, if M is a real symmetric matrix, $\rho(M) = \|M\|$. Finally,

Lemma 2.1. *For all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, we have:*

$$\||x| - |y|\| \leq \|x - y\|.$$

Proof. For detailed proof see Lemma 5 [15]. □

3. The SCQO as an absolute value equation

Recall that the SCQO problem is given by:

$$\min_x \left[f(x) = \frac{1}{2} x^T Q x + x^T b + c \right] \text{ s.t. } x \in \mathbb{S}.$$

Starting from the definition of simplicial cones $\mathbb{S}$ associated with the nonsingular matrix A, the problem (1.1) can be formulated as a quadratic programming problem under positive constraints:

$$\min_y \left[f(y) = \frac{1}{2} y^T A^T Q A y + y^T A^T b + c \right] \text{ s.t. } y \in \mathbb{R}^n_+. \tag{3.1}$$

As the problem (3.1) is convex and the constraints are positive then the optimality conditions of K.K.T are necessary and sufficient and we have, $y \in \mathbb{R}^n_+$ is an optimal solution of problem (3.1) if and only if there exists $z \in \mathbb{R}^n_+$ such that:

$$z - A^T Q A y = A^T b,\ z^T y = 0,\ y \geq 0,\ z \geq 0, \tag{3.2}$$

which is a standard linear complementarity problem (see [8]).
Next, letting $z = |s| - s$ and $y = |s| + s$, then the LCP (3.2) is reformulated as the following absolute value equations (AVE) of type

$$\bar{A}s + \bar{B}|s| = \bar{b}, \tag{3.3}$$

where
$$\bar{A} = A^T QA + I,\ \bar{B} = A^T QA - I,\ \bar{b} = -A^T b.$$
Hence, solving the problem (1.1) is equivalent to solving the AVE (3.3). The following result is needed to guarantee the unique solvability of the AVE.

Theorem 3.1 (Theorem 8 [2]**).** *Assume that $\bar{A}$ is invertible and the matrices $\bar{A}, \bar{B}$ satisfy the following condition, $\left\|\bar{A}^{-1}\bar{B}\right\| < 1$, then the AVE (3.3) has a unique solution for any $\bar{b} \in \mathbb{R}^n$.*

For our case since $\bar{A} = A^T QA + I$ and $\bar{B} = A^T QA - I$ where Q is symmetric positive definite and A is invertible, the condition $\left\|\bar{A}^{-1}\bar{B}\right\| < 1$ of Theorem 3.1, is satisfied. We check this result through the following lemma.

Lemma 3.2. *Let $\bar{A} = A^T QA + I$ and $\bar{B} = A^T QA - I$ such that A is invertible matrix and Q is symmetric positive definite. Then the matrix $\bar{A}$ is invertible and $\left\|\bar{A}^{-1}\bar{B}\right\| < 1$.*

Proof. Because Q is symmetric positive definite and A is invertible, then the matrix $A^T QA$ is symmetric positive definite, hence $\bar{A}$ is symmetric positive definite too, which implies that $\bar{A}$ is invertible. Next, since $A^T QA$ is symmetric positive definite, then $A^T QA$ has positive real eigenvalues denoted by $\lambda_i(A^T QA) := \lambda_i > 0, \forall i = 1, ..., n$. In addition, it is known that the eigenvalues of $\bar{A}$ and $\bar{B}$, are given by $\lambda_i + 1 > 0$ and $\lambda_i - 1$, respectively. Because, $\bar{A}$ and $\bar{B}$ are real symmetric matrices, we then have,
$$\begin{aligned}\left\|\bar{A}^{-1}\bar{B}\right\| &\leq \left\|\bar{A}^{-1}\right\| \left\|\bar{B}\right\| = \rho(\bar{A}^{-1})\rho(\bar{B}) \\ &= \max_i \left(\left| \frac{\lambda_i - 1}{\lambda_i + 1} \right| \right).\end{aligned}$$
As $\lambda_i > 0$, then
$$\left| \frac{\lambda_i - 1}{\lambda_i + 1} \right| < 1.$$
So $\left\|\bar{A}^{-1}\bar{B}\right\| < 1$. This gives the required result. □

Proposition 3.3. *If s^* is the solution of the AVE (3.3) then $(y^*, z^*) = (|s^*| + s^*, |s^*| - s^*)$ is the solution of the LCP (3.2). Consequently, Ay^* is the optimal solution of problem (1.1).*

Proof. Let s^* be the unique solution of the AVE, then
$$\bar{A}s^* + \bar{B}\,|s^*| = \bar{b}.$$
So
$$\begin{aligned}\left(A^T QA + I\right) s^* + (A^T QA - I)\,|s^*| &= -A^T b \\ \Leftrightarrow A^T QA(|s^*| + s^*) + |s^*| - s^* &= -A^T b \\ \Leftrightarrow z^* - A^T QAy^* &= A^T b.\end{aligned}$$
Now, since $y^* = |s^*| + s^* = 2(s^*)^+, z^* = |s^*| - s^* = 2(s^*)^-$, then we have $y^* \geq 0$, $z^* \geq 0$ and $z^{*T} y^* = 0$, hence, the pair (y^*, z^*) is a solution of LCP (3.2). Finally, we deduce that Ay^* is an optimal solution of the SCQO problem. This completes the proof. □

4. Two-steps Picard's fixed point iterative method for SCQO

In this section, we derive a new fixed-point iterative approach for solving the equation (3.3). Let $t = |s|$ then, the AVE (3.3) is equivalent to the following system:

$$\begin{cases} \bar{A}s + \bar{B}t = \bar{b} \\ -|s| + t = 0. \end{cases} \tag{4.1}$$

The latter can be expressed as follows:

$$\begin{pmatrix} \bar{A} & \bar{B} \\ -D(s) & I \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \bar{b} \\ 0 \end{pmatrix}, \tag{4.2}$$

where $D(s)$:=Diag (sign (s)), $s \in \mathbb{R}^n$. Note that the system (4.2) is nonlinear, it is generally impossible to obtain an exact solution. We will therefore be satisfied with an approximated solution. Since the matrix $\bar{A}$ is invertible hence from (4.1) we can obtain the following fixed point equation:

$$\begin{cases} s^* = \bar{A}^{-1}\left(-\bar{B}t^* + \bar{b}\right) \\ t^* = (1-r)\,t^* + r\,|s^*| \end{cases} \tag{4.3}$$

where $r > 0$, is a suitable parameter that we shall specified it later. According to the fixed-point equation, we generate a sequence $(s^{(k)}, t^{(k)})$ converging to the solution of AVE. So the new fixed-point iteration is given by:

$$\begin{cases} s^{(k+1)} = \bar{A}^{-1}\left(-\bar{B}t^{(k)} + \bar{b}\right) \\ t^{(k+1)} = (1-r)\,t^{(k)} + r\left|s^{(k+1)}\right|, k = 0, 1, \ldots \end{cases} \tag{4.4}$$

The details of our algorithm for solving the AVE (3.3) is described in Figure 1.

4.1. Algorithm

Input
An accuracy parameter $\epsilon > 0$;
a parameter r such that $0 < r < \frac{2}{\left\|\bar{A}^{-1}\bar{B}\right\|+1}$;
an initial starting point $t^0 \in \mathbb{R}^n$;
compute $t^1 = (1-r)t^0 + r|s^1|$, $s^1 = \bar{A}^{-1}\left(-\bar{B}t^{(0)} + \bar{b}\right)$; $k := 0$;
  While $\frac{\left\|t^{k+1} - t^k\right\|}{\left\|\bar{b}\right\|} \geq \epsilon$ **do**
    begin
    compute : $\begin{cases} t^{(k+1)} = (1-r)\,t^{(k)} + r\left|s^{(k+1)}\right|, \\ s^{(k+1)} = \bar{A}^{-1}\left(-\bar{B}t^{(k)} + \bar{b}\right) \end{cases}$
    $k := k + 1$;
    end
end

Fig. 1. Algorithm. 4.1

In this section, we give detailed proof for the convergence of Algorithm 4.1.

Theorem 4.1. *Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is an invertible matrix then the sequence $(s^{(k)}, t^{(k)})$ generated by the iterative* *algorithm (4.4) to solve the problem (4.1) is well-defined for any starting point $t^0 \in \mathbb{R}^n$. In addition, if*

$$0 < r < \frac{2}{\left\|\bar{A}^{-1}\bar{B}\right\| + 1},$$

then the sequence $(s^{(k)}, t^{(k)})$ converges linearly to the solution (s^*, t^*) of the nonlinear equation (4.1). Consequently, $A(|s^*| + s^*)$ is the solution of the problem (1.1).

Proof. First, we check that the sequence $(s^{(k)}, t^{(k)})$ is well-defined, it suffices to show that the matrix $\bar{A} = A^T QA + I$ is invertible. This claim was proven by Lemma 3.2. Next, using formula (4.4) and Lemma 2.1, we have, on one hand that

$$\begin{aligned}
\left\|t^{k+1} - t^*\right\| &= \left\|(1-r)\, t^k + r\left|s^{(k+1)}\right| - (1-r)\, t^* + r\left|s^*\right|\right\| \\
&= \left\|(1-r)\left(t^k - t^*\right) + r(\left|s^{(k+1)}\right| - |s^*|)\right\| \\
&\leq |1-r| \left\|(t^k - t^*)\right\| + r\left\|s^{k+1} - s^*\right\|.
\end{aligned}$$

So

$$\begin{aligned}
\left\|s^{k+1} - s^*\right\| &= \left\|\bar{A}^{-1}\left(-\bar{B}t^k + \bar{b}\right) - \bar{A}^{-1}\left(-\bar{B}t^* + \bar{b}\right)\right\| \\
&= \left\|-\bar{A}^{-1}\bar{B}(t^k - t^*)\right\| \leq \left\|\bar{A}^{-1}\bar{B}\right\| \left\|(t^k - t^*)\right\|.
\end{aligned}$$

Therefore

$$\left\|t^{k+1} - t^*\right\| \leq (|1-r| + r\left\|\bar{A}^{-1}\bar{B}\right\|) \left\|(t^k - t^*)\right\|.$$

On the other hand,

$$\begin{aligned}
\left\|s^{k+1} - s^*\right\| &\leq \left\|-\bar{A}^{-1}\bar{B}(t^k - t^*)\right\| \\
&\leq \left\|-\bar{A}^{-1}\bar{B}(t^k - (1-r)t^{k-1} + (1-r)t^{k-1} - t^*)\right\| \\
&\leq \left\|-\bar{A}^{-1}\bar{B}(r\left|s^k\right| + (1-r)t^{k-1} - t^*)\right\|.
\end{aligned}$$

As $|s^*| = t^*$, we find

$$\begin{aligned}
\left\|s^{k+1} - s^*\right\| &\leq \left\|-\bar{A}^{-1}\bar{B}(r\left|s^k\right| - r\left|s^*\right|) - (1-r)\bar{A}^{-1}\bar{B}(t^{k-1} - t^*)\right\| \\
&\leq r\left\|\bar{A}^{-1}\bar{B}\right\| \left\|s^k - s^*\right\| + |1-r| \left\|s^k - s^*\right\| \\
&\leq (|1-r| + r\left\|\bar{A}^{-1}\bar{B}\right\|) \left\|s^k - s^*\right\|.
\end{aligned}$$

The sequence $(s^{(k)}, t^{(k)})$ is convergent if the following condition

$$|1-r| + r\left\|\bar{A}^{-1}\bar{B}\right\| < 1$$

holds. For that we distinguish two cases.
Case 1. If $0 < r \leq 1$, then

$$\begin{aligned}
|1-r| + r\left\|\bar{A}^{-1}\bar{B}\right\| < 1 \quad &\Leftrightarrow \quad 1 - r + r\left\|\bar{A}^{-1}\bar{B}\right\| < 1 \\
&\Leftrightarrow \quad r(\left\|\bar{A}^{-1}\bar{B}\right\| - 1) < 0.
\end{aligned}$$

Since $\left\|\bar{A}^{-1}\bar{B}\right\| < 1$ then,

$$r(\left\|\bar{A}^{-1}\bar{B}\right\| - 1) < 0, \forall\, 0 < r \leq 1.$$

Case 2. If $r \geq 1$, then

$$|1-r| + r\left\|\bar{A}^{-1}\bar{B}\right\| < 1 \quad \Leftrightarrow \quad -1 + r + r\left\|\bar{A}^{-1}\bar{B}\right\| < 1$$
$$\Leftrightarrow \quad r < \frac{2}{\left\|\bar{A}^{-1}\bar{B}\right\| + 1}.$$

Finally, regrouping the two cases, this gives the required result. □

5. Numerical results

In this section, we present numerical results for Algorithm 4.1 by using $\epsilon = 10^{-6}$ and $r = 0.9$. The algorithm has been applied on three examples of SCQO problem. The iterations have been carry out by **MATLAB** R2016a and run on a personal pc with 1.40 GHZ AMD E1-2500 APU Radeon(TM) HD Graphic, 8 GB memory and Windows 10 operating system. The starting point and the unique solution by t^0 and s^*, respectively. The stopping criterion used in our algorithm is the relative residue, i.e.,

$$RES := \frac{\left\|t^{k+1} - t^k\right\|}{\left\|\bar{b}\right\|} \leq 10^{-6}$$

In view of the influence of the initial point on the convergence of our algorithm, different values are used. For each problem, the hypotheses of Theorem 4.1 are checked. In the tables below, the symbols "It" and "CPU" denote the number of iterations produced by the algorithm and the elapsed times, respectively.
Problem 1. Consider the SCQO problem where Q, A and b are given by :

$$Q = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}, A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0.5 & 3 & 0 & 0 & 0 \\ -1 & 0.5 & 3 & 0 & 0 \\ -1 & -1 & 0.5 & 3 & 0 \\ -1 & -1 & -1 & 0.5 & 3 \end{bmatrix}$$

and $b = [-3,\ 1,\ -10,\ -12,\ -2]^T$.
The starting point in this example is taken as:

$$t^0 = [0,\ -1,\ -1,\ 2,\ 1]^T.$$

After 21 iterations, the unique solution s^* of AVE is:

$$s^* = [0.2071,\ -7.6143,\ 0.5262,\ 0.7886,\ -2.2308]^T,$$

and

$$y^* = |s^*| + s^* = [0.4142,\ 0,\ 1.0525,\ 1.5771,\ 0]^T.$$

Therefore, the unique solution of Problem (1.1), is given by:

$$x^* = Ay^* = [1.2426,\ 0.2071,\ 2.7433,\ 4.8435,\ -0.6781]^T.$$

Problem 2. Let the matrices Q, A and the vector b of this example are given by:

$$Q=(q_{ij})=\begin{cases} 4, & \text{for} \quad i=j, \\ \frac{1}{2}, & \text{for} \quad |i-j|=1, i=1,2,\dots n-2, \\ 1, & \text{for} \quad \begin{cases} j=i-2, i=1,2,\dots n, \\ i=j-2, j=1,2,\dots n, \end{cases} \\ 0, & \text{otherwise} \end{cases}$$

$$A=(a_{ij})=\begin{cases} -2, & \text{for} \quad i=j, \\ 4, & \text{for} \quad j=i-1, i=2,\dots n, \\ -1, & \text{for} \quad i=j-1, j=2,\dots n, \\ \frac{1}{2}, & \text{for} \quad j>i+2, i=1,2,\dots n, \\ \frac{1}{5}, & \text{for} \quad i>j+1, j=1,2,\dots n, \end{cases}$$

and

$$b=-2\bar{A}^{-1}(\bar{A}+\bar{B})e.$$

The computational results with different size of n are shown in Table 1. For the initialization of Problem 2, we take different values of t^0.

n		$t^0=[0,\dots,0]^T$	$t^0=[5,\dots,5]^T$	$t^0=[-10,\dots,-10]^T$
10	CPU	$0.04966s$	$0.04552s$	$0.10820s$
	It	17	11	65
	RES	$9.923e-07$	$8.542e-07$	$9.606e-07$
50	CPU	$0.03704s$	$0.03391s$	$0.36887s$
	It	4	3	42
	RES	$9.136e-07$	$9.025e-07$	$9.576e-07$
100	CPU	$0.07339s$	$0.05023s$	$0.26949s$
	It	3	3	22
	RES	$6.032e-07$	$1.508e-07$	$9.365e-07$
1000	CPU	$5.34351s$	$4.42180s$	$5.36816s$
	It	2	1	2
	RES	$5.992e-07$	$3.371e-07$	$7.501e-07$
2000	CPU	$34.17236s$	$33.23488s$	$39.15840s$
	It	1	1	2
	RES	$3.370e-07$	$8.427e-07$	$1.873e-07$

Table 1. Computational results of Problem 2.

An exact solution with different size of n is given by:

$$s^*=[2,\,2,\,\dots,\,2]^T,$$

and

$$y^*=[4,\,4,\,\dots,\,4]^T.$$

An exact solution of problem (1.1) is given by: $x^*=Ay^*$.

Problem 3. The bloc matrices Q,A and the vector b of this example are given by:

$$Q=\begin{bmatrix} Q_{11} & I_n \\ I_n & Q_{22} \end{bmatrix},\ A=\begin{bmatrix} A_{11} & I_n \\ B & A_{11} \end{bmatrix}$$

where

$$Q_{11} = (q_{11})_{ij} = \begin{cases} 6, & \text{for} \quad i = j, \\ -1, & \text{for} \quad |i-j| = 1, i = 1, 2, \dots n, \\ 0, & \text{otherwise.} \end{cases}$$

$$Q_{22} = (q_{22})_{ij} = \begin{cases} 5, & \text{for} \quad i = j, \\ -2, & \text{for} \quad |i-j| = 1, i = 1, 2, \dots n, \\ \frac{1}{4}, & \text{for} \quad \begin{cases} j = i+1, i = 1, 2, \dots n-1, \\ j = i-1, j = 1, 2, \dots n-1, \end{cases} \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{11} = (a_{11})_{ij} = \begin{cases} -2, & \text{for} \quad i = j, \\ -1, & \text{for} \quad j = i-1, i = 3, \dots n, \\ 3, & \text{for} \quad j > i, \quad i = 3, \dots n, \\ 0.5, & \text{otherwise.} \end{cases}$$

$$B = (b_{ij}) = \begin{cases} -1, & \text{for} \quad i = j, \\ 0, & \text{for} \quad \begin{cases} j = i+1, i = 1, 2, \dots n-1, \\ j = i-1, j = 1, 2, \dots n, \end{cases} \\ \frac{1}{n}, & \text{for} \quad \begin{cases} j = i+2, i = 1, 2, \dots n-2, \\ j = i-2, j = 4, \dots n, \end{cases} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$b = [-8, \dots, -8]^T.$$

For the initialization, we take:

$$t^0 = [0, \dots, 0, -1, \dots, -1]^T.$$

The numerical results with different size of n are summarized in Table 2.

n	10	50	100	1000	2000
CPU	$0.14101s$	$1.0919s$	$1.11202s$	$8.37809s$	$35.76440s$
It	122	137	92	6	3
RES	$9.81e-07$	$9.98e-07$	$9.95e-07$	$7.436e-07$	$7.35e-07$

Table 2. Computational results of Problem 3.

For example, if $n = 10$ then,

$$s^* = [0.0984,\ 0.0042, 0.409,\ -1.782,\ 1.6329,\ 0.3763,\ 0.4448, 0.881, 0.7929, 2.7841]^T,$$

and

$$y^* = |s^*| + s^* = [0.1967, 0.0082, 0.8179, 0, 3.2658, 0.7525, 0.8896, 1.7619, 1.586, 5.568]^T.$$

An exact solution of problem (1.1) is given by:

$$x^* = Ay^* = [2.3928, 2.475, 2.3819, 2.206, 2.1049, 2.0303, 3.038, 3.3649, 3.2356, 2.3168]^T.$$

6. Conclusion and remarks

In this paper, a convex quadratic programming problem under simplicial cone constraints were studied, and via its optimality conditions is reduced to finding the unique solution of an absolute value equation AVE. For solving this AVE we applied a new two-steps Picard's iterative fixed point iteration. In particular, the sufficient conditions for the convergence of our algorithm are studied. The obtained numerical results deduced from the testing examples illustrate that the suggested algorithm is efficient and valid to solve the SCQO problems.

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New hybrid conjugate gradient method as a convex combination of PRP and RMIL+ methods

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Abstract. The Conjugate Gradient (CG) method is a powerful iterative approach for solving large-scale minimization problems, characterized by its simplicity, low computation cost and good convergence. In this paper, a new hybrid conjugate gradient HLB method (HLB: Hadji-Laskri-Bechouat) is proposed and analysed for unconstrained optimization. We compute the parameter β_k^{HLB} as a convex combination of the Polak-Ribière-Polyak $\left(\beta_k^{PRP}\right)$ and the Mohd Rivaie-Mustafa Mamat and Abdelrhaman Abashar $\left(\beta_k^{RMIL+}\right)$ i.e. $\beta_k^{HLB} = (1-\theta_k)\,\beta_k^{PRP} + \theta_k \beta_k^{RMIL+}$. By comparing numerically CGHLB with PRP and RMIL+ and by using the Dolan and More CPU performance, we deduce that CGHLB is more efficient.

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Keywords: Unconstrained optimization, hybrid conjugate gradient method, line search, descent property, global convergence.

1. Introduction

Consider the nonlinear unconstrained optimization problem

$$\min_{x\in\mathbb{R}^n} f(x) \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function, bounded from below. The gradient of f is denoted by $g(x)$. To solve this problem, we start from an initial

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point $x_0 \in \mathbb{R}^n$. Nonlinear conjugate gradient methods generate sequences $\{x_k\}$ of the following form:

$$x_{k+1} = x_k + \alpha_k d_k, \qquad k = 0, 1, 2, ..., \tag{1.2}$$

where x_k is the current iterate point and $\alpha_k > 0$ is the step size which is obtained by line search [7].

The iterative formula of the conjugate gradient method is given by (1.2), where d_k is the search direction defined by

$$d_{k+1} = \begin{cases} -g_k & \text{si } k = 1 \\ -g_{k+1} + \beta_k d_k & \text{si } k \geq 2 \end{cases} \tag{1.3}$$

where β_k is a scalar and $g(x)$ denotes $\nabla f(x)$ [10]. If f is a strictly convex quadratic function, namely,

$$f(x) = \frac{1}{2} x^T H x + b^T x, \tag{1.3bis}$$

where H is a positive definite matrix and if α_k is the exact one-dimensional minimizer along the direction d_k, i.e.

$$\alpha_k = \arg\min_{\alpha > 0} \{f(x + \alpha d_k)\} \tag{1.3tris}$$

then (1.2), (1.3), (1.3bis), (1.3tris) is called the linear conjugate gradient method. Otherwise, (1.2), (1.3) is called the nonlinear conjugate gradient method. Conjugate gradient methods can broadly be classified based on the used strategies of the way in which the search direction is updated and the algorithms dealing with the step size minimization along a direction [6]. In [12], a convex combination of LS and FR ([1]) is proposed with a newton descent direction.

The line search in the non linear conjugate gradient methods is often based on the standard Wolfe conditions [23]:

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \rho \alpha_k g_k^t d_k \tag{1.4}$$

$$g_{k+1}^t d_k \geq \delta g_k^t d_k \tag{1.5}$$

where $0 < \rho \leq \delta < 1$.

Conjugate gradient methods differ in their way of defining the scalar parameter β_k. In the literature, there have been proposed several choices for β_k which give rise to distinct conjugate gradient methods [16], [27]. The most well known conjugate gradient methods are the Hestenes–Stiefel (HS) method [17], the Fletcher-Reeves (FR) method [1], [13], the Polak-Ribière-Polyak (PRP) method [20], [19], the Conjugate Descent method(CD) [13], the Liu-Storey (LS) method [18], the Dai-Yuan (DY) method [08], [09], Hager and Zhang (HZ) method [15] and the RMIL+ method [21], [22]. The update parameters of these methods are respectively specified as follows:

$$\beta_k^{HS} = \frac{g_{k+1}^T y_k}{d_k^T y_k}, \ \beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2}, \ \beta_k^{PRP} = \frac{g_{k+1}^T y_k}{\|g_k\|^2}, \ \beta_k^{CD} = -\frac{\|g_{k+1}\|^2}{d_k^T g_k}$$

$$\beta_k^{LS} = -\frac{g_{k+1}^T y_k}{d_k^T g_k}, \ \beta_k^{DY} = \frac{\|g_{k+1}\|^2}{d_k^T y_k}, \ \beta_k^{HZ} = \left(y_k - 2 d_k \frac{\|y_k\|^2}{d_k^T y_k} \right)^T \frac{g_{k+1}}{d_k^T y_k},$$

$$\beta_k^{RMIL+} = \frac{g_{k+1}^T (g_{k+1} - g_k - d_k)}{\|d_k\|^2}.$$

Some of these methods, such as Fletcher and Reeves (FR) [13], Dai and Yuan (DY) [8] and Conjugate Descent (CD) [13] have strong convergence properties, but they may have modest practical performance due to jamming. On the other hand, the methods of Polak and Ribière and Polyak (PRP) [20], Hestenes and Stiefel (HS) [17] or Liu and Story (LS) [18] may not generally be convergent, but they often have better computational performance.

In the process of obtaining more robust and efficient conjugate gradient methods, some researchers suggested the hybrid conjugate gradient algorithm which combined the good features of the methods involve in the hybridization. Even though conjugate gradient improvement using hybridization is a classic deeply investigated problem; it still an attractive topic for the research community due to its contemporary use in numerous prominent disciplines [25].

The first hybrid conjugate gradient method was given by Touati-Ahmed and Storey (1990) [24] to avoid jamming phenomenon.

The researchers were motivated by the works of Andrei [5], [4]; Dai and Yuan [9]; Zhang and Zhou [26]. Their parameter β_k^N is computed as a convex combination of β_k^{FR} and β_k^* other algorithms, i.e.

$$\beta_k^N = (1 - \theta_k)\, \beta_k^{FR} + \theta_k \beta_k^*$$

The Wolfe line search was employed to determine the step length $\alpha_k > 0$ and the new method proved to be more robust numerical wise as compared to FR and other methods. The global convergence was established under some suitable conditions.

In [4] Andrei has proposed a new hybrid conjugate gradient algorithm where the parameter β_k^A is computed as a convex combination of the Polak-Ribière-Polyak and the Dai-Yuan conjugate gradient algorithms i.e.

$$\beta_k^A = (1 - \theta_k)\, \beta_k^{PRP} + \theta_k \beta_k^{DY}$$

and θ_k is presented to satisfy the conjugacy condition

$$\theta_k = \theta_k^{CCOMB} = \frac{(y_k^t g_{k+1})\,(y_k^t s_k) - (y_k^t g_{k+1})\,(g_k^t g_k)}{(y_k^t g_{k+1})\,(y_k^t s_k) - \|g_{k+1}\|^2 \|g_k\|^2}$$

where $s_k = x_{k+1} - x_k$. To satisfy Newton direction he takes

$$\theta_k = \theta_k^{NDOMB} = \frac{(y_k^t g_{k+1} - s_k^t g_{k+1})\,\|g_k\|^2 - (y_k^t g_{k+1})\,(y_k^t s_k)}{\|g_{k+1}\|^2 \|g_k\|^2 - (y_k^t g_{k+1})\,(y_k^t s_k)}$$

but in the combination of HS and DY from Newton direction, he puts

$$\theta_k = \frac{-s_k^t g_{k+1}}{g_k^t g_{k+1}}.$$

On the other hand, from Newton direction with modified secant condition (Hybrid M-Andrei), Andrei has proposed another method

$$\beta_k^{HYBRIDM} = (1 - \theta_k)\, \beta_k^{HS} + \theta_k \beta_k^{DY}$$

where

$$\theta_k = \frac{\left(\frac{\delta\eta_k}{s_k^t s_k} - 1\right) s_k^t g_{k+1} - \frac{y_k^t g_{k+1}}{y_k^t s_k}\delta\eta_k}{g_k^t g_{k+1} + \frac{g_k^t g_{k+1}}{y_k^t s_k}\delta\eta_k}$$

δ is parameter. In [14] Salah Gazi Shareef and Hussein Ageel Khatab have introduced a new hybrid CG method

$$\beta_k^{New} = (1-\theta_k)\,\beta_k^{PRP} + \theta_k\beta_k^{BA}$$

where β_k^{BA} is selected in [2].

Recently Delladji *et al.* [11] proposed a hybridazation of PRP and HZ schemes using the congugacy condition.

In this paper, we present another hybrid CG algorithm noted CGHLB (HLB is an abbreviation to Hadji; Laskri and Bechouat), witch is a convex combination of the PRP ([20]) and RMIL+ ([21]) conjugate gradient algorithms.We are interested to combine these two methods in a hybrid CG algorithm because PRP has good computational properties and RMIL+ has strong convergence properties. In section 2, we introduce our hybrid CG method and prove that it generates descent directions. In Section 3 we present and prove global convergence results. Numerical results and a conclusion are presented in section 4. By comparing numerically CGHLB with PRP and RMIL+ and by using the Dolan and More CPU performance, we deduce that CGHLB is more efficient.

2. HLB conjugate gradient method

The iterates $x_0, x_1, ...$ of the proposed HLB algorithm are computed by means of the recurrence (1.2) where the step size $\alpha_k > 0$ is determined according to the wolfe line search conditions (1.4), (1.5). The directions d_k are generated by the rule:

$$d_k = \begin{cases} -g_0 & \text{if } k = 0 \\ -g_k + \beta_{K-1}^{HLB} d_{k-1} & \text{if } k \geq 1 \end{cases} \tag{2.1}$$

where

$$\beta_k^{HLB} = (1-\theta_k)\,\beta_k^{PRP} + \theta_k\beta_k^{RMIL+}$$

i.e.

$$\beta_k^{HLB} = (1-\theta_k)\,\frac{g_{k+1}^t y_k}{\|g_k\|^2} + \theta_k\frac{g_{k+1}^t\,(g_{k+1} - g_k - d_k)}{\|d_k\|^2} \tag{2.2}$$

HLB is an abbreviation to Hadji; Laskri and Bechouat; θ_k is a scalar parameter which will be determined in a specific way to be described in the following section. Observe that if $\theta_k = 0$ then $\beta_k^{HLB} = \beta_k^{PRP}$ and if $\theta_k = 1$, then $\beta_k^{HLB} = \beta_k^{RMIL+}$. On the other hand if $0 < \theta_k < 1$, then β_k^{HLB} is a convex combination of $\beta_k^{PRP}$and β_k^{RMIL+}. The parameter θ_k is selected in such away that at every iteration the conjugacy condition is satisfied. It can be noted that,

$$d_{k+1} = -g_{k+1} + (1-\theta_k)\,\frac{g_{k+1}^t y_k}{\|g_k\|^2} d_k + \theta_k\frac{g_{k+1}^t\,(g_{k+1} - g_k - d_k)}{\|d_k\|^2} d_k \tag{2.3}$$

so multiply both sides of above equation by y_k and by using the conjugacy condition $\left(d_{k+1}^t y_k = 0\right)$ we have:

$$0 = -g_{k+1}^t y_k + (1-\theta_k)\frac{g_{k+1}^t y_k}{\|g_k\|^2} d_k^t y_k + \theta_k \frac{g_{k+1}^t \left(g_{k+1} - g_k - d_k\right)}{\|d_k\|^2} d_k^t y_k \tag{2,4}$$

After a simple calculation we get

$$\theta_k = \frac{g_{k+1}^t y_k \|g_k\|^2 \|d_k\|^2 - \left(g_{k+1}^t y_k\right)\left(d_k^t y_k\right)\|d_k\|^2}{\left(\left(g_{k+1}^t \left(y_k - d_k\right)\right)\|g_k\|^2 - \left(g_{k+1}^t y_k\right)\|d_k\|^2\right)\left(d_k^t y_k\right)} \tag{2.5}$$

So, to ensure the convergence of this method when the parameter θ_k goes out of interval $]0,1[$, i.e. when $\theta_k \le 0$ or $\theta_k \ge 1$, we prefer to take β_k^{HLB} as following:

$$\beta_k^{HLB} = \begin{cases} (1-\theta_k)\beta_k^{PRP} + \theta_k \beta_k^{RMIL+} & \text{if } 0 < \theta_k < 1 \\ \beta_k^{PRP} & \text{if } \theta_k \le 0 \\ \beta_k^{RMIL+} & \text{if } \theta_k \ge 1 \end{cases} \tag{2.5(bis)}$$

We are now able to present our new algorithm, the Conjugate Gradient CGHLB Algorithm:

CGHLB Algorithm

Step 1: Initialization:

Set $k = 0$, select the initial point $x_o \in \mathbb{R}^n$.select the parameters $0 < \rho \le \delta < 1$, and $\varepsilon > 0$.

Compute $f(x_0)$, and $g_0 = \nabla\ f(x_0)$. Consider $d_0 = -g_0$.

Step 2: Test for continuation of iterations:

If $\|g_k\| \le \varepsilon$ then stop else set. $d_k = -g_k$

Step 3: Line search:

Compute $\alpha_k > 0$ satisfying the Wolfe line search condition (1,4) and (1,5) and update the variables, $x_{k+1} = x_k + \alpha_k d_k$; compute $f(x_{k+1})$, g_{k+1} and $s_k = x_{k+1} - x_k$; $y_k = g_{k+1} - g_k$.

Step 4: θ_k Parameter computation:

If $\left(\left(g_{k+1}^t \left(y_k - d_k\right)\right)\|g_k\|^2 - \left(g_{k+1}^t y_k\right)\|d_k\|^2\right)\left(d_k^t y_k\right) = 0$;

then set $\theta_k = 0$, otherwise, compute θ_k as in (2.5).

Step 5: β_k^{HLB}Conjugate gradient parameter computation:

If $0 < \theta_k < 1$, then compute β_k^{HLB} as in (2.2).

If $\theta_k \ge 1$, then set $\beta_k^{HLB} = \beta_k^{RMIL+}$.

If $\theta_k \le 0$, then set $\beta_k^{HLB} = \beta_k^{PRP}$.

Step 6: Direction computation:

Compute $d_{k+1} = -g_{k+1} + \beta_k^{HLB} d_k$.

Set k=k+1 and go to step 3.

The following theorem shows that our method assures the descent condition, when $0 < \theta_k < 1$.

Theorem 2.1. *In the algorithm (1.2), (1.3) and (2.5) assume that d_k is a descent direction ($g_k^t d_k < 0$), and α_k is determined by the Wolfe line search (1.4); (1.5). If $0 < \theta_k < 1$ then the direction d_{k+1}given by (2.3) is a descent direction.*

Proof. Multiply both sides of (2,3) by g_{k+1} we have:

$$\begin{aligned} g_{k+1}^T d_{k+1} &= -\|g_{k+1}\|^2 + (1-\theta_k)\frac{g_{k+1}^t y_k}{\|g_k\|^2} d_k^t g_{k+1} \\ &\quad +\theta_k \frac{g_{k+1}^t (g_{k+1} - g_k - d_k)}{\|d_k\|^2} d_k^t g_{k+1} \end{aligned}$$

$$\begin{aligned} g_{k+1}^T d_{k+1} &= -(1-\theta_k+\theta_k)\|g_{k+1}\|^2 + (1-\theta_k)\frac{g_{k+1}^t y_k}{\|g_k\|^2} d_k^t g_{k+1} \\ &\quad +\theta_k \frac{g_{k+1}^t (g_{k+1} - g_k - d_k)}{\|d_k\|^2} d_k^t g_{k+1} \end{aligned}$$

$$\begin{aligned} g_{k+1}^T d_{k+1} &= \left[-(1-\theta_k)\|g_{k+1}\|^2 + (1-\theta_k)\frac{g_{k+1}^t y_k}{\|g_k\|^2} d_k^t g_{k+1}\right] \\ &\quad + \left[-(\theta_k)\|g_{k+1}\|^2 + \theta_k \frac{g_{k+1}^t (g_{k+1} - g_k - d_k)}{\|d_k\|^2} d_k^t g_{k+1}\right] \end{aligned}$$

$$\begin{aligned} g_{k+1}^T d_{k+1} &= (1-\theta_k)\left[-\|g_{k+1}\|^2 + \frac{g_{k+1}^t y_k}{\|g_k\|^2} d_k^t g_{k+1}\right] \\ &\quad + (\theta_k)\left[-\|g_{k+1}\|^2 + \frac{g_{k+1}^t (g_{k+1} - g_k - d_k)}{\|d_k\|^2} d_k^t g_{k+1}\right] \end{aligned}$$

since $0 < \theta_k < 1$ then

$$\begin{aligned} g_{k+1}^T d_{k+1} \leq &\left[-\|g_{k+1}\|^2 + \frac{g_{k+1}^t y_k}{\|g_k\|^2} d_k^t g_{k+1}\right] \\ &+ \left[-\|g_{k+1}\|^2 + \frac{g_{k+1}^t (g_{k+1} - g_k - d_k)}{\|d_k\|^2} d_k^t g_{k+1}\right] \end{aligned} \tag{2.6}$$

If the step length α_k is chosen by an exact line search. Then $g_{k+1}^T d_k = 0$.
If the step length α_k is chosen by an inexact line search $\left(g_{k+1}^T d_k \neq 0\right)$ then we have:

$$g_{k+1}^T d_{k+1} < 0$$

because the algorithms of (PRP) and $(RMIL+)$ satisfied the descent property.
The proof is completed. □

3. Global convergence properties

The following assumptions are often needed to prove the convergence of the nonlinear CG:

Assumption 1

The level set $\Omega = \{x \in \mathbb{R}^n / f(x) \leq f(x_0)\}$ *is bounded, where* x_0 *is the starting point.*

Assumption 2

In some neighborhood N *of* Ω*, the objective function is continuously differentiable and its gradient is Lipschitz continuous, namely, there exists a constant* $l > 0$ *such that:*

$$\|g(x) - g(y)\| \leq l \|x - y\| \quad for\ any\ x, y \in N$$

Under these assumptions on f there exists a constant μ such that $\|g(x)\| \leq \mu$, for all $x \in \Omega$.

Lemma 3.1. [28] *Suppose Assumption 1 and 2 hold, and consider any conjugate gradient method (1.2) and (1.3), where* d_k *is a descent direction and* α_k *is obtained by the strong Wolfe line search. If*

$$\sum_{k=1}^{\infty} \frac{1}{\|d_k\|^2} = +\infty \tag{3.1}$$

then

$$\liminf_{k \to \infty} \|g_k\| = 0 \tag{3.2}$$

Assume that the function f is uniformly convex function, i.e. there exists a constant $\Gamma \geq 0$ such that,

$$for\ all\ x, y \in \Omega : (\nabla f(x) - \nabla f(y))^t (x - y) \geq \Gamma \|x - y\|^2 \tag{3.3}$$

and the steplength α_k is given by the strong Wolfe line search.

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \sigma_1 \alpha_k g_k^t d_k \tag{3.4}$$

$$\left|g_{k+1}^t d_k\right| \leq -\sigma_2 g_k^t d_k \tag{3.5}$$

For uniformly convex function which satisfies the above assumptions, we can prove that the norm of d_{k+1} given by (2.3) is bounded above.

Using the above lemma, we obtain the following theorem.

Theorem 3.2. *Suppose that Assumption 1 and 2 hold. Consider the algorithm (1.2), (2.3) and (2.5), where* $0 \leq \theta_k \leq 1$ *and* α_k *is obtained by the strong Wolfe line search (3.4) and (3.5).*

If d_k *tends to zero and there exists non negative constants* η_1 *and* η_2 *such that:*

$$\|g_k\|^2 \geq \eta_1 \|s_k\|^2 \ \ and\ \ \|g_{k+1}\|^2 \leq \eta_2 \|s_k\| \tag{3.6}$$

and f *is uniformly convex function, then*

$$\lim_{k \to \infty} g_k = 0 \tag{3.7}$$

Proof. From (3,3) it follows that

$$y_k^t s_k \geq \Gamma \|s_k\|^2$$

since $0 \leq \theta_k \leq 1$, from uniform convexity and (3.6) we have

$$\left|\beta_k^{HLB}\right| \leq \left|\frac{g_{k+1}^t y_k}{\|g_k\|^2}\right| + \left|\frac{g_{k+1}^t \left(g_{k+1} - g_k - d_k\right)}{\|d_k\|^2}\right|$$

$$\leq \frac{\left|g_{k+1}^t y_k\right|}{\|g_k\|^2} + \frac{\left|g_{k+1}^t y_k\right|}{\|d_k\|^2} + \frac{\left|g_{k+1}^t d_k\right|}{\|d_k\|^2}$$

$$\leq \frac{\|g_{k+1}\| \|y_k\|}{\|g_k\|^2} + \frac{\|g_{k+1}\| \|y_k\|}{\|d_k\|^2} + \frac{\|g_{k+1}\| \|d_k\|}{\|d_k\|^2}$$

from Lipschitz condition

$$\|y_k\| \leq l \|s_k\|$$

$$\left|\beta_k^{HLB}\right| \leq \frac{\|g_{k+1}\| \|y_k\|}{\eta_1 \|s_k\|^2} + \frac{\|g_{k+1}\| \|y_k\|}{\|d_k\|^2} + \frac{\|g_{k+1}\|}{\|d_k\|}$$

$$\leq \frac{\mu l \|s_k\|}{\eta_1 \|s_k\|^2} + \frac{\mu l \|s_k\| \alpha_k^2}{\|s_k\|^2} + \frac{\mu \alpha_k}{\|s_k\|}$$

$$= \frac{\mu l}{\eta_1 \|s_k\|} + \frac{\mu l \alpha_k^2}{\|s_k\|} + \frac{\mu \alpha_k}{\|s_k\|}$$

Hence

$$\|d_{k+1}\| \leq \|g_{k+1}\| + \left|\beta_k^{HLB}\right| \|d_k\|$$

$$\leq \mu + \frac{\mu l \|s_k\|}{\eta_1 \alpha_k \|s_k\|} + \frac{\mu l \|s_k\| \alpha_k^2}{\alpha_k \|s_k\|} + \frac{\mu \alpha_k \|s_k\|}{\alpha_k \|s_k\|}$$

$$= 2\mu + \mu l \alpha_k + \frac{\mu l}{\eta_1 \alpha_k}$$

which implies that (3.1) is true. Therefore, by Lemma 1 we have (3.2), which for uniformly convex functions is equivalent to (3.7). □

4. Numerical results and discussion

In the present numerical experiments, we analyze the efficiency of β^{HLB}, as compared to the classic methods: β^{PRP} and β^{RMIL+}. These comparisons are based on the number of iterations and CPU time per second to reach the optimum. All the comparisons are done with two or three different initial points and different number of variables ranging from 2 to 20000. All variables have been experimented to each function test [3]. For the numerical tests, the strong Wolfe line searches parameters have been experimentally fixed to $\rho = 10^{-3}$ and $\delta = 10^{-4}$. All tests were terminated when the stopping criteria $\|g_k\| \leq \varepsilon$ is fulfilled, where $\varepsilon = 10^{-6}$. When the iteration number exceeds 2000 or the CPU execution time exceeded 500 seconds, the test is considered as failed.

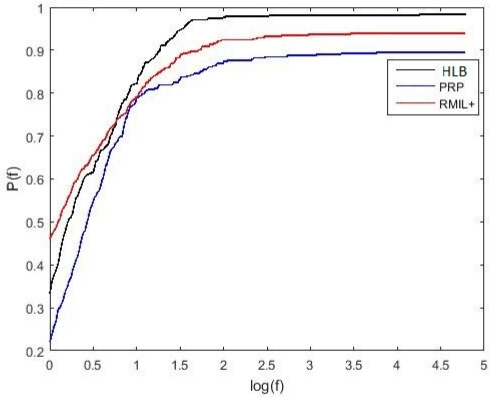

FIGURE 1. Performance Profile based on the CPU time

Figures 1 and 2 show that the method of β^{HLB} is superior when compared to β^{PRP} and β^{RMIL+} with the least duration of CPU time. The highest percentage of successful comparison is with β^{HLB} at 98.34%, followed by β^{RMIL+} with 93.72%. However, the successful rate comparison for β^{PRP} is low at 90.05%. Hence, our method (β^{HLB}) successfully solves the test problems, and it is competitive with the well-known conjugate gradient methods for unconstrained optimization.

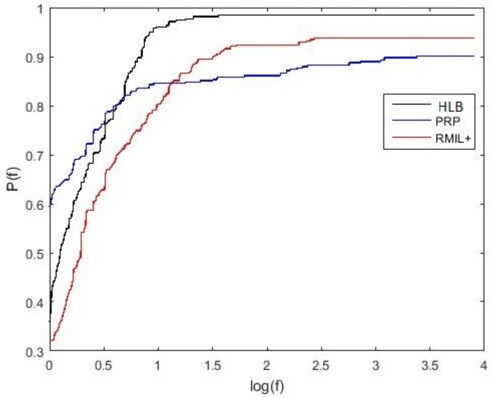

FIGURE 2. Performance Profile based on the iteration number

Table 1. A list of test problems

No.	Function	Dimension	Initial points
01	Alpine 1	$4, 5, 7, 10, 12, 30, 100$	$(1, ..., 1)$
02	Beale	2	$(-1, -1)$; $(0, 0)$; $(1, 1)$
03	Booth	2	$(-1, -1)$; $(1, 1)$; $(3, 3)$
04	Branin	2	$(-1, -1)$; $(0, 0)$; $(1, 1)$
05	Diagonal 1	$2, 4, 6, 8, 10, 20, 100, 200$	$(1, ..., 1)$; $(2, ..., 2)$; $(3, ..., 3)$
06	Diagonal 2	$2, 4, 10, 100, 200, 400, 500, 600, 1000$	$(-1, ..., -1)$; $(0, ..., 0)$; $(1, ..., 1)$
07	Diagonal 4	$1000, 5000, 8000, 10000, 14000, 16000, 20000$	$(2, ..., 2)$; $(5, ..., 5)$; $(10, ..., 10)$
08	Exponential	$2, 4, 6, 8, 10, 12, 14, 15, 16, 20$	$(1, ..., 1)$
09	Griewank	$10, 100, 500, 1000, 2000, 5000, 10000$	$(-2, ..., -2)$; $(2, ..., 2)$
10	Hager	$2, 4, 10, 100, 200, 500, 800, 1000$	$(-1, ..., -1)$; $(0, ..., 0)$
11	Himmelblau	$2, 4, 10, 100, 1000, 5000, 10000, 20000$	$(-5, ..., -5)$; $(5, ..., 5)$
12	Leon	2	$(-0.5, -0.5)$; $(0, 0)$; $(0.5, 0.5)$
13	Matyas	2	$(1, 1)$; $(2, 2)$; $(5, 5)$
14	Penalty	$2, 10, 100, 500, 1000, 2500, 4000, 5000, 10000$	$(-1, ..., -1)$; $(0, ..., 0)$; $(1, ..., 1)$
15	Perquadratic	$2, 4, 8, 10, 20, 50, 200$	$(-5, ..., -5)$; $(3, ..., 3)$; $(5, ..., 5)$
16	Power	$2, 4, 8, 10, 20, 50, 100, 500$	$(-2, ..., -2)$; $(2, ..., 2)$
17	Qing	$2, 10, 100, 200, 300, 400, 500, 1000, 2000$	$(-2, ..., -2)$; $(2, ..., 2)$
18	Quadratic	$2, 10, 100, 200, 500, 750, 1000$	$(2, ..., 2)$; $(4, ..., 4)$
19	Quartic	$2, 4, 10, 100, 200, 500$	$(1, ..., 1)$; $(2, ..., 2)$
20	Rastrigin	$2, 10, 100, 200, 500$	$(-5, ..., -5)$; $(5, ..., 5)$
21	Raydan 1	$2, 4, 10, 20, 50, 80, 90, 100$	$(-2, ..., -2)$; $(2, ..., 2)$
22	Raydan 2	$2, 10, 100, 500, 1000, 2000, 3000$	$(-2, ..., -2)$; $(2, ..., 2)$
23	Rosenbrock	$2, 10, 10, 50, 100, 200, 1000, 2000, 5000, 10000$	$(0, ..., 0)$
24	Schwefel 2. 20	$2, 4, 10, 20$	$(-1, ..., -1)$; $(2, ..., 2)$
25	Schwefel 2. 21	$5, 10, 15, 20$	$(1, ..., 1)$; $(2, ..., 2)$
26	Schwefel 2. 23	$2, 5, 10, 20$	$(-1, ..., -1)$; $(1, ..., 1)$
27	Sphere	$2, 10, 20, 100, 1000, 5000, 20000$	$(-4, ..., -4)$; $(4, ..., 4)$
28	Styblinski	$2, 10, 100, 500, 1000, 2000, 5000$	$(0, ..., 0)$; $(2, ..., 2)$
29	Sumsquares	$2, 10, 20, 100, 300, 500, 1000$	$(5, ..., 5)$; $(10, ..., 10)$

5. Conclusion

Numerous studies have been devoted to develop and improve hybrid conjugate gradient methods. In this paper we have presented a new convex hybridation of the PRP and the RMIL+ conjugate gradient algorithms; HLB. The global convergence of our method is demonstrated for $0 < \theta < 1$. Numerical experiments reveal that our method is reaching the optimum in less iteration number and CPU time comparing to RMIL+ and PRP.

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